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On isomorphisms of Grassmann spaces

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Summary. In this paper an embedding $\phi : P \to P'$ of a projective space $(P, \mathfrak{L})$ into a projective plane $(P', \mathfrak{L}')$ is constructed which satisfies $|L \cap \phi(P)| \geq 2$ for every line $L \in \mathfrak{L}'$. Such an embedding induces a bijection $\beta : \mathfrak{L} \to \mathfrak{L}'$ which maps intersecting lines onto intersecting lines, but not vice versa. This answers an open question about Grassmann spaces.

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1. Introduction

Let $(P, \mathfrak{L})$ denote a projective space with points P and lines $\mathfrak{L}$, and with $\dim P \geq 3$. Then $\mathfrak{L}$ is the point set of the corresponding *Grassmann space* (of index 1). A line of the Grassmann space is the set of all lines of $(P, \mathfrak{L})$ which are contained in a plane of P and containing a common point of P. Hence two points $L, G \in \mathfrak{L}$ of the corresponding Grassmann space lie on a line, if the following binary relation

$$L \sim G \Leftrightarrow L \cap G \neq \emptyset \tag{1}$$

is satisfied. See [4] or [7] for an axiomatic approach for Grassmann spaces and [1] for the *Plücker space*, the classical example of a Grassmann space. Any collineation and, if $\dim P = 3$, any duality of $(P, \mathfrak{L})$ induces an isomorphism of the corresponding Grassmann space, i.e., a bijection which preserves $\sim$ in both directions. W. L. Chow [3] has shown that conversely any isomorphism of the Grassmann space $\mathfrak{L}$ is induced by a collineation or a duality of $(P, \mathfrak{L})$ for $\dim P \in \mathbb{N}$. W. Huang has generalized in [7] Chow's Theorem for Grassmann spaces of an arbitrary index: Any bijection β of $\mathfrak{L}$ for which "$L \sim G$" implies "$\beta(L) \sim \beta(G)$" is an isomorphism of $\mathfrak{L}$. With that result W. Huang answers partly the following question: Let $(\mathfrak{L}, \sim)$ and $(\mathfrak{L}', \sim')$ be two Grassmann spaces. The question is, if a bijection

$$\beta : \mathfrak{L} \to \mathfrak{L}' \qquad \text{with "} L \sim G \Rightarrow \beta(L) \sim' \beta(G) \text{"} \tag{2}$$

is an isomorphism, i.e., $\beta(L) \sim' \beta(G)$ implies $L \sim G$. In a paper of Brauner [2, Satz 2] this property is claimed, but H. Havlicek [6] pointed out a gap in the proof of that result. He proves in Theorem 1 of [6] that for projective spaces $(P, \mathfrak{L}), (P', \mathfrak{L}')$, a bijection $\beta : \mathfrak{L} \to \mathfrak{L}'$ which maps intersecting lines onto intersecting lines is for dim $P \geq 4$ induced by an embedding $\phi : P \to P'$ or for dim $P = 3$ by an embedding of P in P' or in the dual space of P'.

An *embedding* $\phi : P \to P'$ of a linear space $(P, \mathfrak{L})$ into a linear space $(P', \mathfrak{L}')$ is an injective mapping which maps lines of $\mathfrak{L}$ exactly into subsets of lines of $\mathfrak{L}'$, i.e., ϕ maps collinear points into collinear points and non collinear points into non collinear points. An embedding $\phi : P \to P'$ of two projective spaces $(P, \mathfrak{L})$,$(P', \mathfrak{L}')$ which induces a bijection $\beta : \mathfrak{L} \to \mathfrak{L}'$ must have the property that every line $L \in \mathfrak{L}'$ contains the images $\phi(G)$ of a line $G \in \mathfrak{L}$, i.e.,

$$|L \cap \phi(P)| \geq 2 \quad \text{for every line} \quad L \in \mathfrak{L}. \tag{3}$$

If ϕ is surjective, then ϕ is a collineation and hence dim $P =$ dim P'. Therefore if dim $P >$ dim P', then ϕ is not surjective.

In this paper we construct an embedding $\phi : P \to P'$ of a projective space $(P, \mathfrak{L})$ with dim $P \geq 3$ into a projective plane $(P', \mathfrak{L}')$ with the property (3). Clearly (3) implies $|L \cap \phi(P)| \neq 1$ (property (G) of [9]). By Theorem (2.6) of [9], for every embedding $\phi : M \to M'$ of linear spaces $(M, \mathfrak{M}), (M', \mathfrak{M}')$ satisfying dim $M >$ dim M' and property (G), there exist subspaces $P \subset M$ and $P' \subset M'$ satisfying $\dim P > \dim P' = 2$ such that $\phi|_P$ is an embedding of P in the plane P'. Hence we may restrict ourselves to construct an embedding into a projective plane $(P', \mathfrak{L}')$.

The embedding $\phi : P \to P'$ is induced by an embedding f of a vector space (V, K) in a 3–dimensional vector space (L^3, L). We construct f in the following way:

We start with a trivial embedding f_0 of a 3–dimensional vector space (L_0^3, L_0) for a proper field extension L_0 of the field K. Step by step for $i = 0, 1, \ldots$ we extend the vector space V_i to V_{i+1} with $\dim V_{i+1} > \dim V_i$ and simultaneous the field L_i to L_{i+1} which is also a field extension of K with $L_i \subset L_{i+1}$. Also $f_i : V_i \to L_i^3$ is extended to the embedding $f_{i+1} : V_{i+1} \to L_{i+1}^3$. With $V := \bigcup_{i \in \mathbb{N}} V_i$ and $L := \bigcup_{i \in \mathbb{N}} L_i$ we obtain the wanted embedding $f : V \to L^3$. In step I of section 2 the basic construction for one induction step is given and in step II the whole induction step is explained. In step III then $f : V \to L^3$ is defined.

2. Embedding

In this section we give an example of an embedding $\phi : P \to P'$ of a Pappian projective space $(P, \mathfrak{L})$ into a Pappian projective plane $(P', \mathfrak{L}')$ satisfying that $|G \cap \phi(P)| \geq 2$ for every line $G \in \mathfrak{L}'$. For that we construct a mapping f of a vector space (V, K) into a vector space (L^3, L) for a suitable extension field L of a commutative field K.

We recall that the subspaces of the vector spaces (V,K) with dimension 1 and 2, respectively, define the points P and lines $\mathfrak{L}$, respectively, of the projective spaces $(P,\mathfrak{L}) = PG(V,K)$, and that points $x_0 = K\mathfrak{x}_0, \ldots, x_n = K\mathfrak{x}_n$ are independent in $(P,\mathfrak{L})$ if and only if the vectors $\mathfrak{x}_0, \ldots, \mathfrak{x}_n$ are linearly independent in (V,K) (cf. [8]). Hence three points $a = K\mathfrak{a}, b = K\mathfrak{b}, c = K\mathfrak{c}$ are non collinear if and only if rank $(\mathfrak{a},\mathfrak{b},\mathfrak{c}) = 3$.

Let L_i denote a field. For any 2-dimensional subspace E of (L_i^3, L_i) and any subset $W \subset L_i^3$ we denote with

$$\dim' \big(E \cap W\big) \tag{4}$$

the dimension of the subspace of (L_i^3, L_i), which is generated by $E \cap W$. (We mean the dimension relative to the vector space in which E is a subspace.)

First we mention some easy properties of vector spaces (L_i^3, L_i) and (F^3, F) for a field extension F of the field L_i, which we use in the following frequently.

Lemma 2.1. 1. *For any vectors $\mathfrak{a},\mathfrak{b},\mathfrak{c} \in L_i^3$ it holds that $\mathfrak{a},\mathfrak{b},\mathfrak{c}$ are linearly independent in (L_i^3, L_i) if and only if $\mathfrak{a},\mathfrak{b},\mathfrak{c}$ are linearly independent in (F^3, F).*

2. *For a 2-dimensional subspace E of (L_i^3, L_i), $E' := FE + FE$ is the unique determined 2-dimensional vector subspace of (F^3, F) which is generated by E.*

Proof. 1. If $\mathfrak{a},\mathfrak{b},\mathfrak{c}$ are linearly dependent in (L_i^3, L_i), then clearly also in (F^3, F). If $\mathfrak{a},\mathfrak{b},\mathfrak{c}$ are linearly dependent in (F^3, F), then also in (L_i^3, L_i), since else it would follow that $L_i^3 \subset U$ for a proper at most 2-dimensional vector subspace U of F^3 which is generated by $\mathfrak{a},\mathfrak{b},\mathfrak{c}$.

2. For $\mathfrak{a},\mathfrak{b} \in E$ with $E = L_i\mathfrak{a} + L_i\mathfrak{b}$ it follows that $FE + FE = FL_i\mathfrak{a} + FL_i\mathfrak{b} = F\mathfrak{a} + F\mathfrak{b}$. □

In the following for any $i \in \mathbb{N}$, L_i is a field extension of a given field K, (L_i^3, L_i) the 3–dimensional vector space over L_i and let (V_i, K) be a vector space with a basis $\mathfrak{B}_i$. Assume that

$$f_i : V_i \to L_i^3 \tag{5}$$

is an injective mapping satisfying the following two properties:

(α) For $\lambda, \mu \in K$ and $\mathfrak{a},\mathfrak{b} \in V_i$, $f_i(\lambda\mathfrak{a} + \mu\mathfrak{b}) = \lambda f_i(\mathfrak{a}) + \mu f_i(\mathfrak{b})$.

(β) For $\mathfrak{a},\mathfrak{b},\mathfrak{c} \in V_i$, $\mathfrak{a},\mathfrak{b},\mathfrak{c}$ are linearly independent in (V_i, K) only if $f_i(\mathfrak{a}), f_i(\mathfrak{b})$, $f_i(\mathfrak{c})$ are linearly independent in (L_i^3, L_i).

Havlicek calls a mapping f satisfying (α) a *weak semilinear mapping* (cf. [5]). We denote by

$$\mathfrak{E}_i := \{E \subset L_i^3 : \dim E = 2 \text{ and } \dim' \big(E \cap f_i(V_i)\big) = 1\} \quad \text{and} \tag{6}$$

$$\mathfrak{F}_i := \{E \subset L_i^3 : \dim E = 2 \text{ and } \dim' \big(E \cap f_i(V_i)\big) = 0\} \tag{7}$$

the set of all two-dimensional subspaces E of (L_i^3, L_i) for which the line E of the projective space $PG(L_i^3, L_i)$ contains only one point of the via f_i embedded projective space $PG(V_i, K)$, or for which the line E contains no point of the embedded projective space $PG(V_i, K)$, respectively. Furthermore let

$$\mathfrak{G}_i := \mathfrak{E}_i \cup \mathfrak{F}_i. \tag{8}$$

Now let $i \in \mathbb{N}$ for the next two steps be fixed.

I. In a **first step** we choose and define for some fixed $E \in \mathfrak{G}_i$:

- $\begin{cases} \mathfrak{x} \in \big(E \cap f_i(V_i)\big)^*, \text{ i.e., } K\mathfrak{x} = E \cap f_i(V_i) & \text{for } E \in \mathfrak{E}_i \\ \text{any } \mathfrak{x} \in E^* & \text{for } E \in \mathfrak{F}_i \end{cases}$
- $\mathfrak{y} \in E \setminus L_i\mathfrak{x}$, i.e., $\mathfrak{y} \notin f_i(V_i)$ and $E = L_i\mathfrak{x} + L_i\mathfrak{y}$.
- $L_i' := L_i(t)$, the extension field of L_i for a transcendental or algebraic element t over L_i, with degree at least three, i.e., L_i^3 is a subset of $(L_i')^3$.
- (V_i', K), a vector space with the basis $\mathfrak{B}_i' := \mathfrak{B}_i \cup \{\mathfrak{b}\}$ with $\mathfrak{b} \notin V_i$, i.e. $V_i \subset V_i'$ is a proper subspace.

For the subspace E of (L_i^3, L_i) we denote by

$$E' := L_i'E + L_i'E \quad \text{the subspace of } \big((L_i')^3, L_i'\big) \text{ generated by } E. \tag{9}$$

Every vector $\mathfrak{a} \in V_i'$ has the unique representation $\mathfrak{a} = \mathfrak{v} + \lambda\mathfrak{b}$ with $\mathfrak{v} \in V_i$ and $\lambda \in K$. We map $\mathfrak{b}$ to $t\mathfrak{x} + t^2\mathfrak{y}$ and define the following mapping:

$$f_i' : V_i' \to (L_i')^3, \quad \mathfrak{a} = \mathfrak{v} + \lambda\mathfrak{b} \;\mapsto\; f_i'(\mathfrak{a}) := f_i(\mathfrak{v}) + \lambda\big(t\mathfrak{x} + t^2\mathfrak{y}\big) \tag{10}$$

Lemma 2.2. *The mapping f_i' satisfies the properties (α) and (β) and it holds that* $\dim'\big(E' \cap f_i'(V_i')\big) = 2$ *for $E \in \mathfrak{E}_i$ and* $\dim'\big(E' \cap f_i'(V_i')\big) \geq 1$ *for $E \in \mathfrak{F}_i$.*

Proof. (i). Since f_i satisfies (α), by definition also f_i' satisfies (α).

Let $\mathfrak{u}, \mathfrak{v}, \mathfrak{w} \in V_i$. First we show that $\mathfrak{u} + \mathfrak{b}, \mathfrak{v}, \mathfrak{w}$ are linearly independent in (V_i, K) if and only if $f_i'(\mathfrak{u} + \mathfrak{b}), f_i'(\mathfrak{v}), f_i'(\mathfrak{w})$ are linearly independent in $\big((L_i')^3, L_i'\big)$.

(ii). Since $\mathfrak{b} \notin V_i$ and $\mathfrak{u}, \mathfrak{v}, \mathfrak{w} \in V_i$, $\mathfrak{u} + \mathfrak{b}, \mathfrak{v}, \mathfrak{w}$ are linearly independent iff $\mathfrak{v}, \mathfrak{w}$ are linearly independent, i.e, iff $\mathfrak{v}' := f_i'(\mathfrak{v}) = f_i(\mathfrak{v}), \mathfrak{w}' := f_i'(\mathfrak{w}) = f_i(\mathfrak{w})$ are linearly independent in (L_i^3, L_i), since f_i satisfies (β).

(iii). Clearly $f_i'(\mathfrak{u} + \mathfrak{b}) = \mathfrak{u}' + t\mathfrak{x} + t^2\mathfrak{y}, \mathfrak{v}', \mathfrak{w}'$ are linearly dependent in $\big((L_i')^3, L_i'\big)$ iff $\det(\mathfrak{u}' + t\mathfrak{x} + t^2\mathfrak{y}, \mathfrak{v}', \mathfrak{w}') = det(\mathfrak{u}', \mathfrak{v}', \mathfrak{w}') + t\,det(\mathfrak{x}, \mathfrak{v}', \mathfrak{w}') + t^2 det(\mathfrak{y}, \mathfrak{v}', \mathfrak{w}') = 0$. Since $\mathfrak{u}', \mathfrak{x}, \mathfrak{y}, \mathfrak{v}', \mathfrak{w}' \in L_i^3$ and since t has degree at least 3 over L_i, the last equation is equivalent to $det(\mathfrak{u}', \mathfrak{v}', \mathfrak{w}') = 0$, $det(\mathfrak{x}, \mathfrak{v}', \mathfrak{w}') = 0$, and $det(\mathfrak{y}, \mathfrak{v}', \mathfrak{w}') = 0$, i.e., $\mathfrak{u}', \mathfrak{v}', \mathfrak{w}'$, and $\mathfrak{x}, \mathfrak{v}', \mathfrak{w}'$, and $\mathfrak{y}, \mathfrak{v}', \mathfrak{w}'$, respectively, are linearly dependent in (L_i^3, L_i).

Assume that $\mathfrak{v}', \mathfrak{w}'$ are linearly independent, then $\mathfrak{x}, \mathfrak{y} \in L_i\mathfrak{v}' + L_i\mathfrak{w}'$, i.e., $E = L_i\mathfrak{x} + L_i\mathfrak{y} = L_i\mathfrak{v}' + L_i\mathfrak{w}'$ and $\mathfrak{v}', \mathfrak{w}' \in E \cap f_i(V_i)$, a contradiction to $\dim'\big(E \cap f_i(V_i)\big) \leq 1$. Hence $det(\mathfrak{u}' + t\mathfrak{x} + t^2\mathfrak{y}, \mathfrak{v}', \mathfrak{w}') = 0$ iff $\mathfrak{v}', \mathfrak{w}'$ are linearly dependent, i.e. by (ii), iff $\mathfrak{u} + \mathfrak{b}, \mathfrak{v}, \mathfrak{w}$ are linearly dependent in (V_i, K).

Now let $\mathfrak{a} = \mathfrak{u} + \lambda\mathfrak{b}, \mathfrak{b} = \mathfrak{v} + \mu\mathfrak{b}, \mathfrak{c} = \mathfrak{w} + \nu\mathfrak{b} \in V_i'$ with $\mathfrak{u}, \mathfrak{v}, \mathfrak{w} \in V_i$ and $\lambda, \mu, \nu \in K$. Let $\mathfrak{b}' := f_i'(\mathfrak{b}), \mathfrak{u}' := f_i'(\mathfrak{u}), \mathfrak{v}' := f_i'(\mathfrak{v}), \mathfrak{w}' := f_i'(\mathfrak{w})$. We have to show that $f_i'(\mathfrak{a}) = \mathfrak{u}' + \lambda\mathfrak{b}', f_i'(\mathfrak{b}) = \mathfrak{v}' + \mu\mathfrak{b}', f_i'(\mathfrak{c}) = \mathfrak{w}' + \nu\mathfrak{b}'$ are linearly independent in $\big((L_i')^3, L_i'\big)$ if and only if $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ are linearly independent in (V_i, K):

(iv). Since f_i satisfies (β), we can assume that $\lambda \neq 0$ or $\mu \neq 0$ or $\nu \neq 0$. Let $\lambda \neq 0$. Then rank $(\mathfrak{u}' + \lambda\mathfrak{b}', \mathfrak{v}' + \mu\mathfrak{b}', \mathfrak{w}' + \nu\mathfrak{b}') =$ rank $(\lambda^{-1}\mathfrak{u}' + \mathfrak{b}', \lambda\mathfrak{v}' - \mu\mathfrak{u}', \lambda\mathfrak{w}' - \nu\mathfrak{u}')$ with $\lambda\mathfrak{v}' - \mu\mathfrak{u}' = f_i(\lambda\mathfrak{v} - \mu\mathfrak{u}), \lambda\mathfrak{w}' - \nu\mathfrak{u}' = f_i(\lambda\mathfrak{w} - \nu\mathfrak{u}) \in f_i(V_i)$. By (iii) it follows that $\lambda^{-1}\mathfrak{u}' + \mathfrak{b}', \lambda\mathfrak{v}' - \mu\mathfrak{u}', \lambda\mathfrak{w}' - \nu\mathfrak{u}'$ are linearly independent in $\big((L_i')^3, L_i'\big)$ iff $\lambda^{-1}\mathfrak{u} + \mathfrak{b}, \lambda\mathfrak{v} - \mu\mathfrak{u}, \lambda\mathfrak{w} - \nu\mathfrak{u}$ are linearly independent in (V_i, K). Since rank $(\lambda^{-1}\mathfrak{u} + \mathfrak{b}, \lambda\mathfrak{v} - \mu\mathfrak{u}, \lambda\mathfrak{w} - \nu\mathfrak{u}) =$ rank $(\mathfrak{u} + \lambda\mathfrak{b}, \mathfrak{v} + \mu\mathfrak{b}, \mathfrak{w} + \nu\mathfrak{b}) =$ rank $(\mathfrak{a}, \mathfrak{b}, \mathfrak{c})$, the assertion follows.

(v). Because $E' = L_i'\mathfrak{x} + L_i'\mathfrak{y}$, it follows $f_i'(\mathfrak{b}) = t\mathfrak{x} + t^2\mathfrak{y} \in \big(E' \cap f_i'(V_i')\big)$, hence $\dim'\big(E' \cap f_i'(V_i')\big) \geq 1$. Since $\mathfrak{x}, \mathfrak{y}$ are linearly independent in (L_i^3, L_i) and by 2.1 also in $\big((L_i')^3, L_i'\big)$, rank $(t\mathfrak{x} + t^2\mathfrak{y}, \mathfrak{x}) =$ rank $(\mathfrak{y}, \mathfrak{x}) = 2$. If $E \in \mathfrak{E}_i$, then $\mathfrak{x}, t\mathfrak{x} + t^2\mathfrak{y} \in E' \cap f_i'(V_i)$ and hence $\dim'\big(E' \cap f_i'(V_i')\big) = 2$. □

II. In a **second step** we choose and define for **every** $E \in \mathfrak{G}_i$:

- $\begin{cases} \mathfrak{x}_E \in \big(E \cap f_i(V_i)\big)^*, \text{ i.e., } K\mathfrak{x}_E = E \cap f_i(V_i) & \text{for } E \in \mathfrak{E}_i \\ \text{any } \mathfrak{x}_E \in E^* & \text{for } E \in \mathfrak{F}_i \end{cases}$
- $\mathfrak{y}_E \in E \setminus L_i\mathfrak{x}_E$, i.e., $\mathfrak{y}_E \notin f_i(V_i)$ and $E = L_i\mathfrak{x}_E + L_i\mathfrak{y}_E$.
- $L_{i+1} := L_i(T)$ the extension field of L_i with an independent set $T = \{t_E : E \in \mathfrak{G}_i\}$ of transcendental or algebraic elements t_E over L_i such that degree s over $L_i(T \setminus \{s\})$ is at least three for every $s \in T$.
- (V_{i+1}, K), a vector space with a basis $\mathfrak{B}_{i+1} := \mathfrak{B}_i \cup \{\mathfrak{b}_E : E \in \mathfrak{G}_i\}$ with $\mathfrak{b}_E \notin V_i$, i.e. $V_i \subset V_{i+1}$ is a proper subspace.

For every subspace $E \in \mathfrak{G}_i$ of (L_i^3, L_i) we denote with

$$\widehat{E} := L_{i+1}E \quad \text{the by } E \text{ generated subspace of} \quad \big((L_{i+1})^3, L_{i+1}\big). \tag{11}$$

Every vector $\mathfrak{a} \in V_{i+1}$ has the unique representation $\mathfrak{a} = \mathfrak{v} + \sum_{E \in \mathfrak{G}_i} \lambda_E \mathfrak{b}_E$ with $\mathfrak{v} \in V_i$, $\lambda_E \in K$ and $\lambda_E \neq 0$ only for finitely many $E \in \mathfrak{G}_i$. We map $\mathfrak{b}_E$ to $t_E\,\mathfrak{x}_E + t_E^2\,\mathfrak{y}_E$ and define the following mapping:

$$f_{i+1} : \begin{cases} V_{i+1} & \to (L_{i+1})^3 \\ \mathfrak{a} = \mathfrak{v} + \sum\limits_{E \in \mathfrak{G}_i} \lambda_E \mathfrak{b}_E & \mapsto f_{i+1}(\mathfrak{a}) := f_i(\mathfrak{v}) + \sum\limits_{E \in \mathfrak{G}_i} \lambda_E \big(t_E\,\mathfrak{x}_E + t_E^2\,\mathfrak{y}_E\big) \end{cases} \tag{12}$$

Lemma 2.3. *The mapping f_{i+1} satisfies the properties (α) and (β). It holds that* $\dim'\big(\widehat{E} \cap f_{i+1}(V_{i+1})\big) = 2$ *for every $E \in \mathfrak{E}_i$ and* $\dim'\big(\widehat{E} \cap f_{i+1}(V_{i+1})\big) \geq 1$ *for every $E \in \mathfrak{F}_i$. Furthermore $f_{i+1}|_{V_i} = f_i$.*

Proof. (i). Since f_i satisfies (α), by definition also f_{i+1} satisfies (α).

(ii). Let $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in V_{i+1}$. Then there exist a finite number $n \in \mathbb{N}$ and vectors

$\mathfrak{b}_1, \ldots, \mathfrak{b}_n \in \mathfrak{B}_{i+1} \setminus \mathfrak{B}_i$ with

$$\mathfrak{a} = \mathfrak{u} + \sum_{j=1}^{n} \lambda_j \mathfrak{b}_j, \qquad \mathfrak{b} = \mathfrak{v} + \sum_{j=1}^{n} \mu_j \mathfrak{b}_j, \qquad \mathfrak{c} = \mathfrak{w} + \sum_{j=1}^{n} \nu_j \mathfrak{b}_j$$

for $\mathfrak{u}, \mathfrak{v}, \mathfrak{w} \in V_i$ and $\lambda_j, \mu_j, \nu_j \in K$. Since f_i satisfies (β) , $\mathfrak{u}, \mathfrak{v}, \mathfrak{w}$ are linearly independent in (V_i, K) iff $\mathfrak{u}' := f_{i+1}(\mathfrak{u}) = f_i(\mathfrak{u}), \mathfrak{v}' := f_{i+1}(\mathfrak{v}), \mathfrak{w}' := f_{i+1}(\mathfrak{w})$ are linearly independent in (L_{i+1}^3, L_{i+1}). Let denote $\mathfrak{b}_j' := f_{i+1}(\mathfrak{b}_j)$. Now by induction for $k = 1, \ldots, n$, we obtain by 2.2 that

$$\mathfrak{u} + \sum_{j=1}^{k} \lambda_j \mathfrak{b}_j, \quad \mathfrak{v} + \sum_{j=1}^{k} \mu_j \mathfrak{b}_j, \quad \mathfrak{w} + \sum_{j=1}^{k} \nu_j \mathfrak{b}_j \quad \text{are linearly independent iff}$$

$$\mathfrak{u}' + \sum_{j=1}^{k} \lambda_j \mathfrak{b}_j', \quad \mathfrak{v}' + \sum_{j=1}^{k} \mu_j \mathfrak{b}_j', \quad \mathfrak{w}' + \sum_{j=1}^{k} \nu_j \mathfrak{b}_j' \quad \text{are linearly independent.}$$

Hence we summarize that $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ are linearly independent if and only if $f_{i+1}(\mathfrak{a})$, $f_{i+1}(\mathfrak{b}), f_{i+1}(\mathfrak{c})$ are linearly independent, i.e., (β) is satisfied.

(iii). Because $\widehat{E} = L_{i+1}\mathfrak{x} + L_{i+1}\mathfrak{y}$, it follows $f_{i+1}(\mathfrak{b}_E) = t_E\, \mathfrak{x}_E + t_E^2\, \mathfrak{y}_E \in \big(\widehat{E} \cap f_{i+1}(V_{i+1})\big)$, hence $\dim' \big(\widehat{E} \cap f_{i+1}(V_{i+1})\big) \geq 1$. If $E \in \mathfrak{E}_i$, then $\mathfrak{x}_E, t_E\, \mathfrak{x}_E + t_E^2\, \mathfrak{y}_E \in \big(\widehat{E} \cap f_{i+1}(V_{i+1})\big)$ and hence $\dim' \big(\widehat{E} \cap f_{i+1}(V_{i+1})\big) = 2$ (cf. Lemma 2.2).

(iv). By definition of f_{i+1}, it follows that $f_{i+1}|_{V_i} = f_i$. □

III. Now in a **third step** we obtain the wanted result with the following induction.

Let L_0 be a proper extension field of K, $V_0 := K^3$ and

$$f_0 \;:\; V_0 \to L_0^3,\; \mathfrak{x} = (x_0, x_1, x_2) \;\mapsto\; f_0(\mathfrak{x}) := \mathfrak{x} = (x_0, x_1, x_2). \tag{13}$$

Obviously f_0 satisfies $(\alpha), (\beta)$ and since $K \subset L_0$, $\mathfrak{E}_0 := \{E \subset L_0^3 : \dim E = 2$ and $\dim' \big(E \cap f_0(V_0)\big) \leq 1\} \neq \emptyset$.

Using the second step (cf. Lemma 2.3), we construct for $i = 0, 1, 2, \ldots,$:

- an extension field L_{i+1} of L_i,
- a vector space (V_{i+1}, K) with the proper subspace $V_i \subset V_{i+1}$
- and a mapping $f_{i+1} : V_{i+1} \to L_{i+1}^3$ satisfying (α) and (β) with $\dim' \big(\widehat{E} \cap f_{i+1}(V_{i+1})\big) = 2$ for every $E \in \mathfrak{E}_i$ and $\dim' \big(\widehat{E} \cap f_{i+1}(V_{i+1})\big) \geq 1$ for every $E \in \mathfrak{F}_i$.

We define

$$V := \bigcup_{i \in \mathbb{N}} V_i, \qquad L := \bigcup_{i \in \mathbb{N}} L_i. \tag{14}$$

Let for every $\mathfrak{a} \in V$, $n_{\mathfrak{a}} := min\{i \in \mathbb{N} : \mathfrak{a} \in V_i\}$. Then

$$f : V \to L^3,\; \mathfrak{a} \mapsto f(\mathfrak{a}) := f_{n_{\mathfrak{a}}}(\mathfrak{a}) \tag{15}$$

is a mapping with the following properties:

Lemma 2.4. *f is a mapping from the at least 4-dimensional vector space (V, K) into the 3-dimensional vector space (L^3, L) satisfying the properties (α) and (β). For every subspace $E \subset V$ with* $\dim E = 2$ *it holds that* $\dim' \big(E \cap f(V)\big) = 2$.

Proof. It is easy to see that (V, K) is a vector space and L a field extension of K. By the construction of V, clearly $\dim V \geq 4$. For any $\mathfrak{a} \in V$ and every $j \in \mathbb{N}$ with $j \geq n_\mathfrak{a}$ we have by 2.3, $f(\mathfrak{a}) := f_{n_\mathfrak{a}}(\mathfrak{a}) = f_j(\mathfrak{a})$.

For $\lambda, \mu \in K$ and $\mathfrak{a}, \mathfrak{b} \in V$, there is a $n = max\{n_\mathfrak{a}, n_\mathfrak{b}\} \in \mathbb{N}$ with $\lambda\mathfrak{a} + \mu\mathfrak{b} \in V_n$, hence $f(\lambda\mathfrak{a} + \mu\mathfrak{b}) = f_n(\lambda\mathfrak{a} + \mu\mathfrak{b})$ and (α) is satisfied by Lemma 2.3.

Also for $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in V$, there is a $k \in \mathbb{N}$ with $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in V_k$. Again by Lemma 2.3, $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ are linearly independent iff $f(\mathfrak{a}) = f_k(\mathfrak{a}), f(\mathfrak{b}), f(\mathfrak{c})$ are linearly independent. Hence (β) is satisfied for f.

Now let $E \subset L^3$ be a 2-dimensional subspace and $\mathfrak{p}, \mathfrak{q} \in E$ linearly independent. Then there exists an $i \in \mathbb{N}$ with $\mathfrak{p}, \mathfrak{q} \in L_i^3$, hence $\mathfrak{p}, \mathfrak{q} \in E_i := E \cap L_i^3$ and E_i is a subspace of L_i^3 with $\dim E_i = 2$. If $\dim' \big(E_i \cap f_i(V_i)\big) = 2$, then also $\dim' \big(E \cap f(V)\big) = 2$. If $\dim' \big(E_i \cap f_i(V_i)\big) = 1$, then $E_i \in \mathfrak{E}_i$, and by Lemma 2.3 it follows for $\widehat{E}_i = L_{i+1}E_i = E \cap L_{i+1}^3$ that $\dim' \big(\widehat{E}_i \cap f_{i+1}(V_{i+1})\big) = 2$, hence also $\dim' \big(E \cap f(V)\big) = 2$. If $\dim' \big(E_i \cap f_i(V_i)\big) = 0$, then $E_i \in \mathfrak{F}_i$ and by Lemma 2.3 $\dim' \big(\widehat{E}_i \cap f_{i+1}(V_{i+1})\big) \geq 1$. But then in the next induction step $\dim' \big(\widehat{\widehat{E}_i} \cap f_{i+2}(V_{i+2})\big) = 2$ with $\widehat{\widehat{E}_i} = L_{i+1}\widehat{E_i} = L_{i+2}E_i = E \cap L_{i+2}^3$, hence also $\dim' \big(E \cap f(V)\big) = 2$ (cf. 2.1). □

Now Lemma 2.4 implies:

Theorem 2.5. *For every commutative field K there exist a field extension L of K, a projective space $(P, \mathfrak{L}) = PG(V, K)$ and a Pappian projective plane $(P', \mathfrak{L}') = PG(L^3, L)$ with an embedding $\phi : P \to P'$ satisfying $|G \cap \phi(P)| \geq 2$ for every $G \in \mathfrak{L}'$. ϕ is not surjective.*

Proof. We define with the above constructed field extension L of K

$$\phi : P \to P',\ K\mathfrak{a} \mapsto \phi(K\mathfrak{a}) := Lf(\mathfrak{a}), \tag{16}$$

then by (α) and since $K \subset L$, ϕ is well defined and maps collinear points into collinear points. By (β), ϕ maps non collinear points on non collinear points, hence ϕ is an embedding. For every 2-dimensional subspace E of L^3, we have $\dim' \big(E \cap f(V)\big) = 2$ by Lemma 2.4, and by (β), $F := f^{-1}(E \cap f(V))$ is a 2-dimensional subspace of V. That means that the intersection of every line of P' with $\phi(P)$ contains at least two distinct points, hence it is the image of a line of P. Since $\dim V \geq 4$ it follows that $\dim P \geq 3$ and hence that ϕ is not a collineation, i.e., ϕ is not surjective. □

Theorem 2.6. *For every commutative field K there exist a field extension L of K, a projective space $(P, \mathfrak{L}) = PG(V, K)$ and a Pappian projective plane $(P', \mathfrak{L}') = PG(L^3, L)$ with a bijection $\beta : \mathfrak{L} \to \mathfrak{L}'$ which maps any two distinct lines onto intersecting lines. There exist in particular lines with an empty intersection which are mapped under β into intersecting lines.*

Proof. Let $\phi : P \to P'$ be the embedding of Theorem 2.5, and let for a line $G \in \mathfrak{L}$, $\widehat{G}$ denote the line of $\mathfrak{L}'$ which is generated by $\phi(G)$. We define

$$\beta : \mathfrak{L} \to \mathfrak{L}', G \mapsto \widehat{G}. \tag{17}$$

Since ϕ is an embedding, β is injective, and since $|L \cap \phi(P)| \geq 2$, i.e., $L \cap \phi(P) \in \{\phi(G) : G \in \mathfrak{L}\}$ for every $L \in \mathfrak{L}'$, β is surjective. Because $(P', \mathfrak{L}')$ is a projective plane, for $G_1, G_2 \in \mathfrak{L}$ every two lines $\widehat{G}_1, \widehat{G}_2$ have a non empty intersection, and because dim $P' \geq 3$ there are lines $G_1, G_2 \in \mathfrak{L}$ with an empty intersection. □

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