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Webs with Rotation and Reflection Properties and their Relations with Certain Loops

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Abstract. In a web one can define in a natural way reflections in generators and a kind of rotations in points. The structure of the webs and the corresponding loops in which some of these maps are automorphisms will be studied in a synthetic way.

1 Introduction and Notations

Let $\mathcal{W} := (\mathcal{P}, \mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3)$ be a web, i.e. a nonempty set $\mathcal{P}$ and three subsets $\mathcal{G}_i$ of the power set $\mathfrak{P}(\mathcal{P})$ of $\mathcal{P}$ such that

W1 $\forall x \in \mathcal{P},\ \forall i \in \{1,2,3\},\ \exists_1\ [x]_i \in \mathcal{G}_i : x \in [x]_i,$

W2 $\forall i, j \in \{1,2,3\}, i \neq j,\ \forall A \in \mathcal{G}_i,\ \forall B \in \mathcal{G}_j : |A \cap B| = 1,$

and let $\mathcal{G} := \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3$. In this paper we let $\left(\begin{smallmatrix} 1 & 2 & 3 \\ i & j & k \end{smallmatrix}\right) \in S_3$, i.e. $\{1,2,3\} = \{i, j, k\}$, unless specified otherwise, and for $x, y \in \mathcal{P}$ let

$$x \,\square_{ij}\, y := [x]_i \cap [y]_j \quad \text{and} \quad x \,\square\, y := x \,\square_{12}\, y.$$

Each automorphism $\alpha \in \operatorname{Aut}(\mathcal{W}) := \operatorname{Aut}(\mathcal{P}, \mathcal{G})$, respectively $\beta \in \operatorname{Aut}(\mathcal{P}, \mathcal{G}_i \cup \mathcal{G}_j)$ induces a permutation $\alpha' \in S_3$, respectively $\beta' \in S_2$ defined by $\alpha([x]_l) = [\alpha(x)]_{\alpha'(l)}$, $l \in \{1,2,3\}$, respectively $\beta([x]_l) = [\beta(x)]_{\beta'(l)}$, $l \in \{i, j\}$. We set

$$\operatorname{Aut}(\mathcal{P}, \mathcal{G}_i \cup \mathcal{G}_j)^+ := \{\beta \in \operatorname{Aut}(\mathcal{P}, \mathcal{G}_i \cup \mathcal{G}_j) \mid \beta' = \mathrm{id}\},$$
$$\operatorname{Aut}(\mathcal{P}, \mathcal{G}_i \cup \mathcal{G}_j)^- := \{\beta \in \operatorname{Aut}(\mathcal{P}, \mathcal{G}_i \cup \mathcal{G}_j) \mid \beta' = (i, j)\}$$

and

$$\operatorname{Aut}(\mathcal{W})^+ := \{\alpha \in \operatorname{Aut}(\mathcal{W}) \mid \alpha' = \mathrm{id}\}.$$

By **W1** and **W2** each automorphism $\alpha \in \operatorname{Aut}(\mathcal{W})$ is completely determined by its action on two generators $A \in \mathcal{G}_i$, $B \in \mathcal{G}_j$ or on one generator and the corresponding permutation $\alpha' \in S_3$.

A subset $C \subset \mathcal{P}$ is called an *i-chain* if for each $Y \in \mathcal{G}_j \cup \mathcal{G}_k$ the intersection $Y \cap C$ consists of a single point. Let $\mathfrak{C}_i := \{C \in \mathfrak{P}(\mathcal{P}) \mid \forall X \in \mathcal{G} \backslash \mathcal{G}_i : |C \cap X| = 1\}$

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be the set of all i-chains, then $\mathcal{G}_i \subset \mathfrak{C}_i$. To each chain $C \in \mathfrak{C}_i$, in particular to each generator there corresponds a *reflection*

$$\widetilde{C} : \mathcal{P} \to \mathcal{P}; x \mapsto \big[[x]_j \cap C\big]_k \cap \big[[x]_k \cap C\big]_j,$$

i.e. an involution of the set $\mathcal{P}$ fixing exactly the points of C and interchanging the generators of $\mathcal{G}_j$ and $\mathcal{G}_k$, i.e. $\widetilde{C} \in \mathrm{Aut}(\mathcal{P}, \mathcal{G}_j \cup \mathcal{G}_k)$. $\widetilde{C}$ will be called a *chain reflection* (of type i). Then by [5, (2.3, 2.4, 2.5)] we have:

1.1 $\forall C, D \in \mathfrak{C}_i$, $\forall 0 \in \mathcal{P}$:

(1) $\widetilde{C} \circ \widetilde{C} = \mathrm{id}$, $\mathrm{Fix}\,\widetilde{C} = C$, $\widetilde{C}(D) \in \mathfrak{C}_i$, *and* $\widetilde{\widetilde{C}(D)} = \widetilde{C} \circ \widetilde{D} \circ \widetilde{C}$,

(2) $\forall X \in \mathcal{G}_j : \widetilde{C}(X) \in \mathcal{G}_k$, $\forall Y \in \mathcal{G}_k : \widetilde{C}(Y) \in \mathcal{G}_j$, *hence* $\widetilde{C} \in \mathrm{Aut}(\mathcal{P}, \mathcal{G}_j \cup \mathcal{G}_k)^-$,

(3) $\widetilde{[0]}_i([0]_j) = [0]_k$,

(4) $\forall \alpha \in \mathrm{Aut}(\mathcal{P}, \mathcal{G}_j \cup \mathcal{G}_k)^-$ *with* $\alpha^2 = \mathrm{id} : \mathrm{Fix}\,\alpha \in \mathfrak{C}$ *and* $\widetilde{\mathrm{Fix}\,\alpha} = \alpha$.

Two chains $A, B \in \mathfrak{C}_i$ are called *orthogonal* and denoted by $A \perp B$ if $A \neq B$ and $\widetilde{A}(B) = B$. We set $A^\perp := \{X \in \mathfrak{C}_i \mid X \perp A\}$. By (1.1.3), $\widetilde{[0]}_j \circ \widetilde{[0]}_i \circ \widetilde{[0]}_k([0]_i) = [0]_i$. If $\widetilde{[0]}_j \circ \widetilde{[0]}_i \circ \widetilde{[0]}_k|_{[0]_i} = \mathrm{id}|_{[0]_i}$, then $\mathcal{W}$ is called *hexagonal* with respect to 0.

Besides the reflections in chains or generators one can define in a web $\mathcal{W}$ in a natural way a kind of local maps called *rotations*: If γ is one of the cyclic permutations (132) or (123) $\in A_3$ and if $0 \in \mathcal{P}$ is a point of the web $\mathcal{W}$, let γ_0 be the permutation of the set $[0] := [0]_1 \cup [0]_2 \cup [0]_3$ defined by

$\gamma_0(x) = [0]_{\gamma(i)} \cap [x]_{\gamma^{-1}(i)}$ for $x \in [0]_i$, $i \in \{1, 2, 3\}$.

1.2 *Let* $\gamma :=$ (132) *or* (123) $\in A_3$. *Then for each* $0 \in \mathcal{P}$, *the group* $< \gamma_0 >$ *generated by the rotation* γ_0 *is a subgroup of the permutation group* $Sym([0])$ *and we have:*

(1) $\mathrm{Fix}(\gamma_0) = \{0\}$ *and* $(\gamma_0)^2(x) = \widetilde{[0]_{\gamma(i)}}(x)$ *for* $x \in [0]_i$, $i \in \{1, 2, 3\}$,

$$\big((132)_0\big)^2(x) = \begin{cases} \widetilde{[0]}_3(x) = \widetilde{[0]}_3 \circ \widetilde{[0]}_1(x) = \widetilde{[0]}_2 \circ \widetilde{[0]}_3(x) & \text{if } x \in [0]_1 \\ \widetilde{[0]}_1(x) = \widetilde{[0]}_1 \circ \widetilde{[0]}_2(x) = \widetilde{[0]}_3 \circ \widetilde{[0]}_1(x) & \text{if } x \in [0]_2 \\ \widetilde{[0]}_2(x) = \widetilde{[0]}_2 \circ \widetilde{[0]}_3(x) = \widetilde{[0]}_1 \circ \widetilde{[0]}_2(x) & \text{if } x \in [0]_3 \end{cases},$$

(2) $(\gamma_0)^6 = \mathrm{id} \Leftrightarrow \mathcal{W}$ *is hexagonal with respect to* 0,

(3) $(\gamma_0)^6 = \mathrm{id} \Leftrightarrow (\gamma_0)^2 = \begin{cases} \widetilde{[0]}_1 \circ \widetilde{[0]}_2|_{[0]} & \text{if } \gamma = (132) \\ \widetilde{[0]}_2 \circ \widetilde{[0]}_1|_{[0]} & \text{if } \gamma = (123) \end{cases}.$

Note that $(\gamma_0)^{-1} = (\gamma^{-1})_0$ and that γ_0 induces on the set $\{[0]_1, [0]_2, [0]_3\}$ the permutation γ, while $(\gamma_0)^2$ induces γ^{-1}, and $(\gamma_0)^3$ and $(\gamma_0)^6$ the identity. If there is an automorphism ω of the web $\mathcal{W}$ such that the restriction $\omega|_{[0]}$ coincides with one of the maps $(\gamma_0)^6$, $(\gamma_0)^3$, $(\gamma_0)^2$ or γ_0, then ω is unique by (2.8) and is called the *automorphic extension* and we say that the point 0 is *n-extendable* for $n \in \{1, 2, 3, 6\}$ if $\omega|_{[0]} = (\gamma_0)^{\frac{6}{n}}$. Clearly if $\omega|_{[0]} = \gamma_0$, then $\omega^2|_{[0]} = (\gamma_0)^2$ and $\omega^3|_{[0]} = (\gamma_0)^3$, i.e. if $0 \in \mathcal{P}$ is 6-extendable, then also 1-, 2- and 3-extendable, and if 0 is 2- and 3-extendable, then also 6-extendable. If for $n \in \{1, 2, 3, 6\}$ a point $0 \in \mathcal{P}$ is n-extendable and if moreover $(\gamma_0)^6 = \mathrm{id}$, i.e. the web $\mathcal{W}$ is hexagonal with respect to 0 by the above (1.2), then 0 is called *n-rotational*. Moreover if each point of a web $\mathcal{W}$ is n-extendable, respectively n-rotational, then $\mathcal{W}$ is called a *n-extendable*, respectively *n-rotational web*.

From the definition it follows that γ_0 and $(\gamma_0)^2$ are distinct from the identity on $[0]$, but for $(\gamma_0)^3$ this is not provable. We call 0 a *characteristic 2 point* if $(\gamma_0)^3 = \mathrm{id}$. Clearly if 0 is a characteristic 2 point, then $\mathcal{W}$ is hexagonal with respect to 0 and 0 is trivially 2-rotational and if moreover 0 is 3-rotational, then also 6-rotational.

Remarks. 1. We have $(\gamma_0)^{-1} = (\gamma_0)^2 \Leftrightarrow (\gamma_0)^3 = \mathrm{id} \Leftrightarrow \gamma_0$ has the order 3 $\Rightarrow \mathcal{W}$ is hexagonal with respect to 0 in the following way: Let $(a, b, c, d) \in \mathcal{P}^4$. If $[a]_i = [b]_i$, $[b]_j = [c]_j$, $[c]_i = [d]_i$ and $[d]_j = [a]_j$, then (a, b, c, d) is called a *parallelogram*, if moreover $[a]_k = [c]_k$, respectively $[a]_k = [c]_k$ and $[b]_k = [d]_k$, then (a, b, c, d) is called a parallelogram with a diagonal, respectively a Fano-parallelogram. Hexagonal with respect to 0 means: Each parallelogram (a, b, c, d) with $0 \in \{a, b, c, d\}$ and with a diagonal is a Fano-parallelogram.

2. If $\mathcal{W}$ is hexagonal with respect to 0, then γ_0 has either the order 6 or 3. If γ_0 has order 6, then $(\gamma_0)^3$ is involutory and we call $(\gamma_0)^3$ a *quasi-reflection* with respect to 0.

The map γ_0 belongs to the group $\Sigma_0 := \{\sigma \in Sym[0] \mid \exists \sigma' \in S_3 : \forall i \in \{1, 2, 3\} : \sigma([0]_i) = [0]_{\sigma'(i)}\}$. For all $\sigma \in \Sigma_0$ and $i,\ j \in \{1, 2, 3\}$ we associate in a natural way permutations $\sigma_{i,j}$ and σ_i of the whole point set $\mathcal{P}$ (cf. (2.6)).

Remark. In [1, Chap.V] BELOUSOV considers the maps $(\gamma_0)_1$ and $((\gamma_0)^3)_1$, and calls $(\gamma_0)_1$ rotation if $(\gamma_0)_1 \in \mathrm{Aut}(\mathcal{W})$ and $((\gamma_0)^3)_1$ central symmetry if $((\gamma_0)^3)_1 \in \mathrm{Aut}(\mathcal{W})$.

We recall, fixing a point $0 \in \mathcal{P}$, the set $E := [0]_k$ can be turned via a *loop-derivation* $L(\mathcal{W}; 0; i, j)$ into a loop $(E, +)$, where 0 is the neutral element of $(E, +)$. The binary operation "+" of $L(\mathcal{W}; 0; i, j)$ is given by (cf. [5])

$$+ : \begin{cases} E \times E \to E \\ (x, y) \to x + y := \Big[\big[[0]_i \cap [x]_j\big]_k \cap [y]_i\Big]_j \cap E. \end{cases}$$

A bijection $\square_{ij} : E \times E \to E;\ (x, y) \to x \,\square_{ij}\, y$ is a coordinatization of $\mathcal{W}$.

For a loop $(E, +)$ we define: $\forall a \in E$, let $a^+ : E \to E; x \mapsto a + x$, $-a := (a^+)^{-1}(0)$, and $\sim a$ the solution of $x + a = 0$, i.e. $-a$ is the right inverse of a and $\sim a$ the left inverse. Instead of $a + (-b)$ we write $a - b$. Also $\forall a, b \in E$, let

$$\delta_{a,b} := \big((a + b)^+\big)^{-1} \circ a^+ \circ b^+ \quad \text{and} \quad \nu : E \to E; x \mapsto -x.$$

A loop $(E, +)$ is called a *Bol-loop* if for all $a, b \in E$, $a^+ \circ b^+ \circ a^+ = (a + (b + a))^+$. And a Bol-loop is called a *Bruck-loop* if $\nu \in \mathrm{Aut}(E, +)$ and *Moufang-loop* if ν is an antiautomorphism (cf. [8] p. 4, 5). Between a web $\mathcal{W}$ and a loop derivation $(E, +) := L(\mathcal{W}; 0; i, j)$ we have by (2.5) the connection: $\widetilde{[0]_j} \in \mathrm{Aut}(\mathcal{W}) \Leftrightarrow \nu$ is an antiautomorphism of $(E, +)$.

We will need the following configurational statements: By the *local Thomsen condition* $(T, 0; i, j)$ we understand:

(T, 0; i, j) $\forall x \in [0]_i, \forall y \in [0]_j : [0]_k \cap \left[[x]_k \cap [y]_i\right]_j \cap \left[[y]_k \cap [x]_j\right]_i \neq \phi$,

and by the *local Reidemeister condition* $(R, 0, i)$

(R, 0, i) Let $p, q \in [0]_i$, $p_k := [0]_k \cap [p]_j$, $q_j := [0]_j \cap [q]_k$, $p' := [p_k]_i \cap [q]_j$ and $q' := [q_j]_i \cap [p]_k$, then $\left[[p']_k \cap [0]_j\right]_i = \left[[q']_j \cap [0]_k\right]_i$, where $\{i, j, k\} = \{1, 2, 3\}$.

$(T, 0; i, j)$, respectively $(R, 0, i)$ is the specialization of the Thomsen-**TH**, respectively Reidemeister-**RE** condition (cf. [7, p. 80, 81], [11], [12]).

We will show in Theorem (3.2) that a point 0 of $\mathcal{W}$ is 3-extendable if there is an $i \in \{1, 2, 3\}$ such that $(R, 0, i)$ is satisfied. Hence by (1.2) we obtain that 0 is 3-rotational if $\mathcal{W}$ is hexagonal with respect to 0 and $(R, 0, i)$ is satisfied. For the loop derivation $(E, +) := L(\mathcal{W}; 0; i, j)$ this is equivalent to: $\forall a, b \in E : -(a+b)+a = -b$ (cf. (3.3)). Then we make the assumption that the web $\mathcal{W}$ contains 2-extendable, respectively 2-rotational points. By (2.7), (3.6), the remarks on (3.6) and the results of [5, section 6] we have:

1.3 Theorem. *For a point 0 of a web $\mathcal{W}$ the following properties* (1), (2), (3) *and* (4), *respectively* (5) *and* (6), *respectively* (7) *and* (8) *are equivalent:*

(1) *0 is a 2-extendable point,*
(2) *For $(E, +) = L(\mathcal{W}; 0; i, j)$, $i, j \in \{1, 2, 3\}$, $i \neq j$ the map ν is an automorphism of $(E, +)$ (cf. (3.6)),*
(3) *The bend-configuration* $BE(0; \mathrm{id})$ *closes, i.e.* $\forall p \in \mathcal{P}$:

$$\left[\left[[[p]_1 \cap [0]_3]_2 \cap [0]_1\right]_3 \cap [0]_2\right]_1 \cap \left[\left[[[p]_2 \cap [0]_1]_3 \cap [0]_2\right]_1 \cap [0]_3\right]_2 \cap \left[\left[[[p]_3 \cap [0]_2]_1 \cap [0]_3\right]_2 \cap [0]_1\right]_3 \neq \phi$$

(cf. [5, (6.3)]),
(4) *If $\mathcal{W}$ is coordinatized by $(E, +) = L(\mathcal{W}; 0; i, j)$, then the map*

$$x \,\square_{ij}\, y \mapsto (-x) \,\square_{ij}\, (-y) \in \mathrm{Aut}(\mathcal{W}).$$

(5) *0 is a 2-rotational point,*
(6) *For $(E, +) = L(\mathcal{W}; 0; i, j)$, ν is an involutory automorphism of $(E, +)$,*
(7) *0 is a characteristic 2 point,*
(8) *For $(E, +) := L(\mathcal{W}; 0; i, j)$, ν is the identity.*

By Theorem (3.8) a point 0 is then 6-extendable if the local Thomsen condition $(T, 0; i, j)$ is valid and on the algebraic side that means: $\forall a, b \in (E, +) : b + (a - b) = a$, i.e. $(E, +)$ is a crossed-inverse loop by the terminology of Bruck [2]. The similar result of (3.8) is in [1, Theorem 5.4]. Moreover we also obtain Theorem (3.9) which states the property of the orbit $[p]^i := \{\widetilde{X}(p) \mid X \in \mathcal{G}_i\}$ of a point $p \in \mathcal{P}$, $i \in \{1, 2, 3\}$, related to a 2-rotational point, if there is an $E \in \mathcal{G}_i$ with $\widetilde{E} \in \mathrm{Aut}(\mathcal{W})$.

2 Local Symmetries

In our web $\mathcal{W} = (\mathcal{P}, \mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3)$, we consider two generators $A, B \in \mathcal{G}_i$ of the same type. Then by (1.1) $\widetilde{A} \in \mathrm{Aut}(\mathcal{P}, \mathcal{G}_j \cup \mathcal{G}_k)$ if $\{j, k\} = \{1, 2, 3\} \setminus \{i\}$ and $\widetilde{A}(B) \in \mathfrak{C}_i$, but in general $\widetilde{A}(B)$ is not contained in $\mathcal{G}_i$. Therefore

2.1 *Let* $A \in \mathcal{G}_i$*, then*

(1) $\widetilde{A} \in \mathrm{Aut}(\mathcal{P}, \mathcal{G}) \Leftrightarrow \widetilde{A} \in \mathrm{Aut}(\mathcal{P}, \mathcal{G}_i)$,

(2) *If* $\widetilde{A} \in \mathrm{Aut}(\mathcal{P}, \mathcal{G}_i)$*, then* $\mathcal{W}$ *is hexagonal with respect to each point* $x \in A$.

For the following let $0 \in \mathcal{P}$ be fixed, let $(E, +) := L(\mathcal{W}; 0; 1, 2)$ be the corresponding loop, and note that $\Box := \Box_{12}$. Then we have:

2.2 $\mathcal{W}$ *is hexagonal with respect to* 0 *if and only if* $\forall a \in E : -a + a = 0$.

Now we consider the map $\widetilde{E}$. Since $x = b\Box c$, we obtain $\widetilde{E}(x) = c\Box b$, $\widetilde{E}(0\Box a) = a\Box 0$ and we have: $\widetilde{E} \in \mathrm{Aut}(\mathcal{P}, \mathcal{G}_3) \Leftrightarrow c\Box b \in [a\Box 0]_3 \Leftrightarrow b = {\sim}a + c = {\sim}a + (a + b)$.

For $b = -a$ we obtain $-a = {\sim}a$ and so:

2.3 $\widetilde{E} \in \mathrm{Aut}(\mathcal{P}, \mathcal{G}_3) \Leftrightarrow \forall a, b \in E : b = -a + (a + b)$*, i.e.* $\forall a \in E : \delta_{-a,a} = \mathrm{id}$.

Next we consider the map $\phi := \widetilde{[0]_1}$. By $x = [0\Box c]_2 \cap [0\Box a]_3, b = [0\Box b]_2 \cap [0]_3$ we have: $\phi(x) = [0\Box c]_3 \cap [0\Box a]_2 = [0\Box c]_3 \cap [a]_2$ and $\phi(b) = [0\Box b]_3 \cap [0]_2 = (-b)\Box 0$. Since $[b]_1 = [x]_1$, we have: $\phi \in \mathrm{Aut}(\mathcal{P}, \mathcal{G}_1) \Leftrightarrow \phi(x) = [0\Box c]_3 \cap [a]_2 \in [-b\Box 0]_1 = [-b]_1 \Leftrightarrow a = c + (-b) = (a + b) - b$. This gives us the result:

2.4 $\widetilde{[0]_1} \in \mathrm{Aut}(\mathcal{P}, \mathcal{G}_1) \Leftrightarrow \forall a, b \in E : (a + b) - b = a$.

Finally we study the map $\psi := \widetilde{[0]_2}$. Here we have $x = [-a\Box 0]_3 \cap [b\Box 0]_1$, hence $\psi(x) = [-a\Box 0]_1 \cap [b\Box 0]_3 = [-a]_1 \cap [0\Box {\sim}b]_3$ and $c = [0]_3 \cap [c\Box 0]_1$, hence $\psi(c) = [0]_1 \cap [c\Box 0]_3 = [0]_1 \cap [0\Box {\sim}c]_3$. This implies $[{\sim}b + (-a)]_2 = [\psi(x)]_2$ and $[{\sim}c]_2 = [{\sim}(a + b)]_2 = [\psi(c)]_2 = [\psi(a + b)]_2$. Since $[x]_2 = [a + b]_2$, we have: $\psi \in \mathrm{Aut}(\mathcal{P}, \mathcal{G}_2) \Leftrightarrow [{\sim}b + (-a)]_2 = [{\sim}(a + b)]_2 \Leftrightarrow {\sim}b + (-a) = {\sim}(a + b)$. For $b = 0$ we obtain $-a = {\sim}a$ and so:

2.5 $\widetilde{[0]_2} \in \mathrm{Aut}(\mathcal{P}, \mathcal{G}_2) \Leftrightarrow \forall a, b \in E : -(a + b) = -b + (-a)$*, i.e.* ν *is an antiautomorphism.*

Now we study the following question: Let σ be one of the permutations of the set $\{(\gamma_0)^i, \widetilde{[0]_i}|_{[0]} \mid i \in \{1, 2, 3\}\}$ of $Sym[0]$, where γ_0 was introduced in Section 1 (cf. (1.2)). When is σ extendable to an automorphism $\overline{\sigma}$ of $(\mathcal{P}, \mathcal{G})$? These maps belong to the class Σ_0 of permutations of the set $[0] = [0]_1 \cup [0]_2 \cup [0]_3$ defined by $\Sigma_0 := \{\sigma \in Sym[0] \mid \forall i \in \{1, 2, 3\} : \sigma([0]_i) \in \{[0]_1, [0]_2, [0]_3\}\}$. To each $\sigma \in \Sigma_0$ there correspond the following permutations σ_{ij} and σ_i of the set $\mathcal{P}$: Let $\sigma' \in S_3$ be defined by $\sigma([0]_i) = [0]_{\sigma'(i)}$ and let $i' := \sigma'(i)$ for $i \in \{1, 2, 3\}$, then

$$\sigma_{i,j} : \begin{cases} \mathcal{P} = [0]_i \Box_{ji} [0]_j \to \mathcal{P} = [0]_{i'} \Box_{j'i'} [0]_{j'} \\ x \Box_{ji} y \to \sigma(x) \Box_{j'i'} \sigma(y) \end{cases},$$

and

$$\sigma_i : \begin{cases} \mathcal{P} = [0]_i \Box_{jk} [0]_i \to \mathcal{P} = [0]_{i'} \Box_{j'k'} [0]_{i'} \\ x \Box_{jk} y \to \sigma(x) \Box_{j'k'} \sigma(y) \end{cases}.$$

Note that σ_3 is equal to the extension $\overline{\sigma}$ considered in the Extension Theorem (2.8) of [5]. Then σ_{ij}, respectively σ_i is an extension of $\sigma|_{[0]_i \cup [0]_j}$, respectively $\sigma|_{[0]_i}$ onto $\mathcal{P}$. And the generators $X \in \mathcal{G}_i$, $Y \in \mathcal{G}_j$, $Z \in \mathcal{G}_k$ have the images: If $x_j := X \cap$

$[0]_j$, $x_k := X \cap [0]_k$, $y_i := Y \cap [0]_i$, $y_k := Y \cap [0]_k$, $z_i := Z \cap [0]_i$, $z_j := Z \cap [0]_j$, then

$$X = [x_j]_i = [0]_i \Box_{ji} x_j,$$
$$Y = [y_i]_j = y_i \Box_{ji} [0]_j = [0]_j \Box_{ij} y_i = y_i \Box_{jk} [0]_i,$$
$$Z = [z_i]_k = z_i \Box_{kj} [0]_i, = [0]_i \Box_{jk} z_i,$$

and so

$$\sigma_{ij}(X) = [0]_{i'} \Box_{j'i'} \sigma(x_j) = [\sigma(x_j)]_{i'} \in \mathcal{G}_{i'},$$
$$\sigma_{ij}(Y) = \sigma(y_i) \Box_{j'i'} [0]_{j'} = [\sigma(y_i)]_{j'} \in \mathcal{G}_{j'},$$
$$\sigma_i(Y) = \sigma_i(y_i \Box_{jk} [0]_i) = \sigma(y_i) \Box_{j'k'} [0]_{i'} = [\sigma(y_i)]_{j'} \in \mathcal{G}_{j'},$$
$$\sigma_i(Z) = [0]_{i'} \Box_{j'k'} \sigma(z_i) = [\sigma(z_i)]_{k'} \in \mathcal{G}_{k'}.$$

So we have proved:

2.6 *For all $\sigma \in \Sigma_0$, the maps σ_{ij} and σ_k are isomorphisms from $(\mathcal{P}, \mathcal{G}_i, \mathcal{G}_j)$ onto $(\mathcal{P}, \mathcal{G}_{i'}, \mathcal{G}_{j'})$ with $\sigma_{ij}|_{[0]_i \cup [0]_j} = \sigma|_{[0]_i \cup [0]_j}$, $\sigma_{ij} = \sigma_{ji}$ and $\sigma_k|_{[0]_k} = \sigma|_{[0]_k}$.*

Next we discuss when for $\sigma \in \Sigma_0$ the maps σ_{ij} and σ_k are automorphisms of $(\mathcal{P}, \mathcal{G})$.

Definition. Let $\sigma \in \Sigma_0$ and let $(x, y) \in [0]_i \times [0]_j$. Then σ is called *(i,j)-faithful* if $x \Box_{ji} y \in [0]_k \Rightarrow \sigma(x) \Box_{j'i'} \sigma(y) \in [0]_{k'}$, and *k-faithful* if $[x]_k = [y]_k \Rightarrow [\sigma(x)]_{k'} = [\sigma(y)]_{k'}$.

2.7 *For $\sigma \in \Sigma_0$ we have*

(1) *$\sigma_{ij}([0]_k) \in \mathcal{G} \Leftrightarrow \sigma$ is (i,j)-faithful,*
(2) *$\sigma_{ij}|_{\mathcal{G}_i} = \sigma_{ik}|_{\mathcal{G}_i} \Leftrightarrow \sigma$ is i-faithful,*
(3) *$\sigma_{ij}|_{[0]} = \sigma \Leftrightarrow \sigma$ is (i,j)- and i-faithful $\Leftrightarrow \sigma$ is i- and j-faithful $\Leftrightarrow \sigma_{ij}|_{\mathcal{G}_i} = \sigma_{ik}|_{\mathcal{G}_i} \wedge \sigma_{ji}|_{\mathcal{G}_j} = \sigma_{jk}|_{\mathcal{G}_j}$,*
(4) *$\sigma_{ij} \in \mathrm{Aut}(\mathcal{P}, \mathcal{G}) \Rightarrow \sigma$ is (i,j)-faithful and k-faithful,*
(5) *$\sigma_{ij} \in \mathrm{Aut}(\mathcal{P}, \mathcal{G}) \Leftrightarrow \forall a, b \in [0]_i \times [0]_j$ and $c \in [0]_i$ with $[c]_k = [a \Box_{ji} b]_k$: $\sigma(a) \Box_{j'i'} \sigma(b) \in [\sigma(c)]_{k'}$,*
(6) *$\sigma_{ij} = \sigma_{ik} \Leftrightarrow \sigma_{ij} \in \mathrm{Aut}(\mathcal{P}, \mathcal{G}) \wedge \sigma$ is i-faithful,*
(7) *$\sigma_k|_{[0]_i} = \sigma|_{[0]_i} \Leftrightarrow \sigma$ is j-faithful $\Leftrightarrow \sigma_k|_{[0]_k \cup [0]_i} = \sigma_i|_{[0]_k \cup [0]_i} = \sigma|_{[0]_k \cup [0]_i}$,*
(8) *$\sigma_k|_{[0]} = \sigma \Leftrightarrow \sigma$ is i- and j-faithful $\Leftrightarrow \sigma_{ij} = \sigma_k$,*
(9) *$\sigma_k \in \mathrm{Aut}(\mathcal{P}, \mathcal{G}) \Leftrightarrow$ For $a, b \in [0]_k$, $c \in [0]_i$ with $[c]_k = [a \Box_{ij} b]_k$: $[\sigma(c)]_{k'} = [\sigma(a) \Box_{i'j'} \sigma(b)]_{k'}$,*
(10) *$\sigma_k = \sigma_j \Leftrightarrow \sigma_k \in \mathrm{Aut}(\mathcal{P}, \mathcal{G}) \wedge \sigma$ is i-faithful,*
(11) *$\sigma_{ij} = \sigma_j \Leftrightarrow$ For $a, b, c \in [0]_j$ with $[c]_k \cap [0]_i \in [[a]_k \cap [b]_i]_j : \sigma([c]_k \cap [0]_i) \in \left[[\sigma(a)]_{k'} \cap [\sigma(b)]_{i'}\right]_{j'}$.*

Proof. (2) Let $X \in \mathcal{G}_i$ and $x_j := X \cap [0]_j$, $x_k := X \cap [0]_k$, hence $X = [x_j]_i = [x_k]_i$. Then by (2.6), $\sigma_{ij}(X) = [\sigma(x_j)]_{i'}$, and $\sigma_{ik}(X) = [\sigma(x_k)]_{i'}$. Hence $\sigma_{ij}(X) = \sigma_{ik}(X) \Leftrightarrow [\sigma(x_j)]_{i'} = [\sigma(x_k)]_{i'} \Leftrightarrow \sigma$ is i-faithful.
(3) By (2.6), $\sigma_{ij}|_{[0]} = \sigma \Leftrightarrow \sigma_{ij}|_{[0]_k} = \sigma|_{[0]_k}$, which implies $\sigma_{ij}([0]_k) = [0]_{k'}$, hence by (1), σ is (i, j)-faithful. Now we assume that σ is (i, j)-faithful. Let

$x \in [0]_k, x_j := [x]_i \cap [0]_j$ hence $x = [0]_k \cap [x_j]_i$ and $[x]_i = [x_j]_i$. Then $\sigma_{ij}(x) = \sigma_{ij}([0]_k) \cap [\sigma(x_j)]_{i'} = [0]_{k'} \cap [\sigma(x_j)]_{i'} = \sigma(x) \Leftrightarrow [\sigma(x_j)]_{i'} = [\sigma(x)]_{i'}$, hence σ is i-faithful, and clearly if σ is (i, j)-faithful and i-faithful, then also j-faithful. Finally let σ be i- and j-faithful and let $x_i \in [0]_i$, $x_j \in [0]_j$ such that $x := x_j \Box_{ij} x_i \in [0]_k$. Then $[x]_i = [x_j]_i$ and $[x]_j = [x_i]_j$, and by the i- and j-faithful assumption, $[\sigma(x)]_{i'} = [\sigma(x_j)]_{i'}$ and $[\sigma(x)]_{j'} = [\sigma(x_i)]_{j'}$. Therefore $[0]_{k'} \ni \sigma(x) = [\sigma(x)]_{i'} \cap [\sigma(x)]_{j'} = [\sigma(x_j)]_{i'} \cap [\sigma(x_i)]_{j'} = \sigma(x_j)\Box_{i'j'}\sigma(x_i)$, i.e. σ is (i, j)-faithful. And the last equivalence is immediate from the above (2).

(6) Let $\sigma_{ij} = \sigma_{ik}$, then by (2.6), $\sigma_{ij} \in \mathrm{Aut}(\mathcal{P}, \mathcal{G})$ and by (2.7.2), σ is i-faithful. Now let $\sigma_{ij} \in \mathrm{Aut}(\mathcal{P}, \mathcal{G})$ and σ is i-faithful, and let $p \in \mathcal{P}$, $p_j := [p]_i \cap [0]_j$, $p_i := [p]_j \cap [0]_i$ and $q_k := [p]_i \cap [0]_k$, $q_i := [p]_k \cap [0]_i$, hence $p = [p_i]_j \cap [p_j]_i = [q_i]_k \cap [q_k]_i$ and $[p_j]_i = [q_k]_i$. Since σ is i-faithful, $[\sigma(p_j)]_{i'} = [\sigma(q_k)]_{i'}$ and since $\sigma_{ij} \in \mathrm{Aut}(\mathcal{P}, \mathcal{G})$, $\sigma_{ij}(p) = [\sigma(p_i)]_{j'} \cap [\sigma(p_j)]_{i'} \in [\sigma(q_i)]_{k'}$, thus $\sigma_{ij}(p) = [\sigma(q_i)]_{k'} \cap [\sigma(p_j)]_{i'} = [\sigma(q_i)]_{k'} \cap [\sigma(q_k)]_{i'} = \sigma_{ik}(p)$.

(10) Let $\sigma_k = \sigma_j$, then $\sigma_k|_{[0]_j} = \sigma_j|_{[0]_j} \overset{(2.6)}{=} \sigma|_{[0]_j}$, i.e. σ is i-faithful by (2.7.7), and $\sigma_k \in \mathrm{Aut}(\mathcal{P}, \mathcal{G})$ by (2.6). Now let $\sigma_k \in \mathrm{Aut}(\mathcal{P}, \mathcal{G})$ and σ i-faithful and let $p \in \mathcal{P}, a_k := [p]_i \cap [0]_k$, $a_j := [p]_i \cap [0]_j$, $b_k := [p]_j \cap [0]_k$ and $c_j := [p]_k \cap [0]_j$. Then $p = [a_k]_i \cap [b_k]_j = [a_j]_i \cap [c_j]_k$ and $[a_k]_i = [a_j]_i$. Since σ is i-faithful, $[\sigma(a_k)]_{i'} = [\sigma(a_j)]_{i'}$, and since $\sigma_k \in \mathrm{Aut}(\mathcal{P}, \mathcal{G})$, $\sigma_k(p) = [\sigma_k(c_j)]_{k'} \cap [\sigma(a_j)]_{i'}$. Therefore $\sigma_j(p) = [\sigma(a_j)]_{i'} \cap [\sigma(c_j)]_{k'} = \sigma_k(p)$ if $\sigma(c_j) = \sigma_k(c_j)$, i.e. if $\sigma|_{[0]_j} = \sigma_k|_{[0]_j}$, i.e. by (2.7.7) if σ is i-faithful. $\square$

2.8 For each $\gamma \in A_3 \setminus \{\mathrm{id}\}$ and each $0 \in \mathcal{P}$ the map $\gamma_0 (\in \Sigma_0)$ is 1-, 2- and 3-faithful by definition and so are the maps $(\gamma_0)^2$ and $(\gamma_0)^3$. Therefore by (2.7.8) for $\phi = (\gamma_0)^i \in \Sigma_0$ we have $\phi = \phi_k|_{[0]}$ with $\phi_{ij} = \phi_k$ and moreover by (2.7.10) if $\phi_k \in \mathrm{Aut}(\mathcal{W})$, then $\phi_i = \phi_{ij}$ for all $i, j \in \{1, 2, 3\}, i \neq j$, hence the automorphic extension $\phi_k (= \phi_i = \phi_{ij})$ of ϕ is unique.

3 Extensions of Local Symmetries and Some Orbits

Firstly we discuss when the maps γ_0, $(\gamma_0)^2$ and $(\gamma_0)^3$ are extendable. For the convenience, we set the maps $0_6 := \gamma_0$, $0_3 := (\gamma_0)^2$ and $\tilde{0} := (\gamma_0)^3$, where γ is taken as $(132) \in A_3$. We consider the maps $\widetilde{[0]}_i|_{[0]}$ and 0_3 in the following:

3.1 *Let $\gamma = (132) \in S_3$, $i \in \{1, 2, 3\}$ and $j := \gamma(i)$, $k := \gamma(j)$. Then:*

(1) $(\widetilde{[0]}_i|_{[0]})_i = \widetilde{[0]}_i$,

(2) $0_3|_{[0]_i} = \widetilde{[0]}_j \circ \widetilde{[0]}_k|_{[0]_i}$ *and* $(0_3)_i = (\widetilde{[0]}_j \circ \widetilde{[0]}_k|_{[0]})_i$,

(3) $(0_3)_i = \widetilde{[0]}_j \circ \widetilde{[0]}_k \Leftrightarrow \widetilde{[0]}_k \in \mathrm{Aut}(\mathcal{P}, \mathcal{G}_k) \Rightarrow (0_3)^3 = \mathrm{id}_{[0]}$,

(4) *Let* $\widetilde{[0]}_k \in \mathrm{Aut}(\mathcal{P}, \mathcal{G}_k)$, *hence* $(0_3)_i = \widetilde{[0]}_j \circ \widetilde{[0]}_k$ *by (3). Then* $(0_3)_i$ *is an isomorphism from* $(\mathcal{P}, \mathcal{G}_j \cup \mathcal{G}_k)$ *onto* $(\mathcal{P}, \mathcal{G}_j \cup \mathcal{G}_k)$, *and* $(0_3)_i \in \mathrm{Aut}(\mathcal{P}, \mathcal{G}) \Leftrightarrow \widetilde{[0]}_j \in \mathrm{Aut}(\mathcal{P}, \mathcal{G}_j)$,

(5) *If* $\widetilde{[0]}_i$, $\widetilde{[0]}_j \in \mathrm{Aut}(\mathcal{P}, \mathcal{G})$, *then* $\widetilde{[0]}_k \in \mathrm{Aut}(\mathcal{P}, \mathcal{G}_k)$ *and* $(0_3)_i = (0_3)_j = (0_3)_k \in \mathrm{Aut}(\mathcal{P}, \mathcal{G})$, *i.e.* $\mathcal{W}$ *is 3-rotational with respect to* 0.

Proof. (1) To $\widetilde{[0]}_i|_{[0]}$ there corresponds the transposition (j,k). Therefore, if $p \in \mathcal{P}$, $p_j := [p]_j \cap [0]_i$ and $p_k := [p]_k \cap [0]_i$, (hence $p = p_j \square_{jk}\, p_k = [p_j]_j \cap [p_k]_k$), then $(\widetilde{[0]}_i|_{[0]})_i(p) = [p_j]_k \cap [p_k]_j = \widetilde{[0]}_i(p)$.
(2) and (3) By (1.2.1), $0_3([0]_i) = {0_6}^2([0]_i) = \widetilde{[0]}_j \circ \widetilde{[0]}_k([0]_i)$ and to 0_3 and to $\phi := \widetilde{[0]}_j \circ \widetilde{[0]}_k|_{[0]}$ there corresponds the same permutation σ. Thus $(0_3)_i = (\widetilde{[0]}_j \circ \widetilde{[0]}_k|_{[0]})_i = (\phi)_i$ and (2) is completely proved. Now $(\phi)_i(p) = [\phi(p_j)]_k \cap [\phi(p_k)]_i = [\widetilde{[0]}_k(p_j)]_k \cap [\widetilde{[0]}_k(p_k)]_i$ and $\widetilde{[0]}_j \circ \widetilde{[0]}_k(p) = [\widetilde{[0]}_k(p_j)]_k \cap [q]_i$, where $q := [0]_j \cap [\widetilde{[0]}_k(p)]_k$. We have

$$(\phi)_i(p) = \widetilde{[0]}_j \circ \widetilde{[0]}_k(p) \Leftrightarrow q = \widetilde{[0]}_k(p_k) \Leftrightarrow \widetilde{[0]}_k(p) \in [\widetilde{[0]}_k(p_k)]_k.$$

Since $p \in \mathcal{P}$ is arbitrary and $[p]_k = [p_k]_k$, we obtain the equivalence of (3). If $\widetilde{[0]}_k \in \mathrm{Aut}(\mathcal{P}, \mathcal{G}_k)$, then $\mathcal{W}$ is hexagonal with respect to 0 by (2.1) and we obtain $(0_3)^3 = \mathrm{id}_{[0]}$ by (1.2.2).
(4) Since $\widetilde{[0]}_i(\mathcal{G}_j) = \mathcal{G}_k$ and $\widetilde{[0]}_i(\mathcal{G}_k) = \mathcal{G}_j$, we obtain $(0_3)_i(\mathcal{G}_j) = \mathcal{G}_k$ and by assumption $(0_3)_i(\mathcal{G}_k) = \widetilde{[0]}_j \circ \widetilde{[0]}_k(\mathcal{G}_k) = \widetilde{[0]}_j(\mathcal{G}_k) = \mathcal{G}_i$. Therefore, $(0_3)_i \in \mathrm{Aut}(\mathcal{P}, \mathcal{G}) \Leftrightarrow (0_3)_i(\mathcal{G}_i) = \widetilde{[0]}_j \circ \widetilde{[0]}_k(\mathcal{G}_i) = \widetilde{[0]}_j(\mathcal{G}_j) = \mathcal{G}_j \Leftrightarrow \widetilde{[0]}_j \in \mathrm{Aut}(\mathcal{P}, \mathcal{G}_j)$.
(5) $\widetilde{[0]}_k \overset{(1.1)}{=} \widetilde{\widetilde{[0]}_i([0]_j)} \overset{(1.1)}{=} \widetilde{[0]}_i \circ \widetilde{[0]}_j \circ \widetilde{[0]}_i \in \mathrm{Aut}(\mathcal{P}, \mathcal{G})$. □

3.2 Theorem. *The following statements are equivalent:*

(1) $(R, 0, 1)$ *is satisfied,*
(2) $(0_3)_1 \in \mathrm{Aut}(\mathcal{P}, \mathcal{G})$,
(3) $\forall a, b \in E : \sim(a+b) + a = \sim b$,
(4) $\forall i \in \{1,2,3\} : (R, 0, i)$ *is satisfied,*
(5) $\forall i \in \{1,2,3\} : (0_3)_i \in \mathrm{Aut}(\mathcal{P}, \mathcal{G})$,
(6) 0 *is a 3-extendable point.*

Proof. Let $X \in \mathcal{G}_1$ and $q' \in X$ be given. We set $E := [0]_3$ and construct:

$$p := [q']_3 \cap [0]_1, \qquad a := p_3 := [p]_2 \cap E, \qquad b := X \cap E,$$
$$q_2 := X \cap [0]_2, \qquad q := [q_2]_3 \cap [0]_1.$$

Then $[q']_2 \cap [0]_3 = a + b$, $q = 0\square(\sim b)$, $p' := [p_3]_1 \cap [q]_2 = a\square(\sim b)$ and $r := [p']_3 \cap [0]_2 = [a\square(\sim b)]_3 \cap [0]_2$ and we have: $[a+b]_1 = \big[[q']_2 \cap [0]_3\big]_1 = [r]_1 \Leftrightarrow r = (a+b)\square 0 \Leftrightarrow \sim(a+b) + a = \sim b$. Moreover, let $s := [q']_2 \cap [0]_1$. Then $q' = [p]_3 \cap [s]_2$, $(0_3)(p) = a\square 0$, $0_3(s) = (a+b)\square 0$, hence $(0_3)_1(q') = [a\square 0]_1 \cap [(a+b)\square 0]_3 = [a]_1 \cap [(a+b)\square 0]_3$ and $q_2 := [q]_3 \cap [0]_2$, $(0_3)(q) = \sim b\square 0$, $0_3(0) = 0$, hence $(0_3)_1(q_2) = [\sim b\square 0]_1 \cap [0]_3 = \sim b$. Therefore $(0_3)_1(X) \in \mathcal{G}_2 \Leftrightarrow (0_3)_1(X) = [\sim b]_2 \Leftrightarrow p' = [a]_1 \cap [(a+b)\square 0]_3$, so $(R, 0, 1) \Leftrightarrow \sim(a+b) + a = \sim b$. Since $(R, 0, i) \Leftrightarrow (R, 0, j)$, we have the equivalence of statements (1), (2), (3), (4) and (5). Since 0_3 is 1-, 2- and 3-faithful, we have $(0_3)_i|_{[0]} = 0_3$, and so if (2) is assumed, then 0 is a 3-extendable point. □

If $\mathcal{W}$ is 3-extendable and hexagonal with respect to 0, then it is 3-rotational with respect to 0. So by (2.2) and (3.2) we obtain:

3.3 *The following statements are equivalent:*

(1) *$\mathcal{W}$ is 3-rotational with respect to 0,*
(2) *For* $(E,+) := L(\mathcal{W};0;i,j) : \forall a,b \in E : -(a+b)+a = -b.$

3.4 *Under the assumption*

(0) $\forall a,b \in E : \sim(a+b)+a = \sim b,$

the following three assertions are equivalent:

(1) $\forall a,b \in E : (a+b)-b = a,$
(2) $\forall a,b \in E : -(a+b) = -b+(-a),$
(3) $\forall a,b \in E : -a+(a+b) = b.$

Proof. (1) $\Rightarrow$ (2) If we set $a := \sim b$, then we obtain $-b = \sim b$ from (1), and (0) resumes the form $-(a+b)+a = -b$. We substitute in (1) a to $-b$, b to $-a$ and obtain $(-b-a)+a = -b$. Together with (0) this implies $-(a+b) = -b+(-a)$.
Similarly applying $-b = \sim b$, we see that (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (1). □

From (2.3), (2.4), (2.5), (3.2) and (3.4) it follows

3.5 *Let $\mathcal{W}$ be 3-extendable with respect to* 0, *then*

(1) *The three assertions* $\widetilde{[0]}_i \in \mathrm{Aut}(\mathcal{P}, \mathcal{G}_i)$ *for* $i \in \{1,2,3\}$ *are equivalent,*
(2) *If there is an* $i \in \{1,2,3\}$ *such that* $\widetilde{[0]}_i \in \mathrm{Aut}(\mathcal{P}, \mathcal{G})$, *then $\mathcal{W}$ is 3-rotational with respect to 0 (cf. (1) and (3.1)).*

Next we study the turn $\widetilde{0} = (0_6)^3$.

3.6 *The following statements are equivalent:*

(1) $\exists i \in \{1,2,3\} : (\widetilde{0})_i \in \mathrm{Aut}(\mathcal{P}, \mathcal{G}),$
(2) $\forall i \in \{1,2,3\} : (\widetilde{0})_i \in \mathrm{Aut}(\mathcal{P}, \mathcal{G}),$
(3) $\forall x \in \mathcal{P} : \widetilde{[0]}_2 \circ \widetilde{[0]}_1 \circ \widetilde{[0]}_3([x]_1) \cap \widetilde{[0]}_3 \circ \widetilde{[0]}_2 \circ \widetilde{[0]}_1([x]_2) \cap \widetilde{[0]}_1 \circ \widetilde{[0]}_3 \circ \widetilde{[0]}_2([x]_3) \neq \phi,$
(4) *For* $(E,+) := L(\mathcal{W};0;i,j)$, $\nu \in \mathrm{Aut}(E,+)$.

Proof. The permutation corresponding to $\widetilde{0}$ in S_3 is the identity. Therefore $(\widetilde{0})_1 \in \mathrm{Aut}(\mathcal{P}, \mathcal{G}) \Leftrightarrow (\widetilde{0})_1 \in \mathrm{Aut}(\mathcal{P}, \mathcal{G}_1) \Leftrightarrow \forall x \in \mathcal{P}$ if $x_2 := [x]_2 \cap [0]_1$, $x_3 := [x]_3 \cap [0]_1$, $y := [x]_1 \cap [0]_3$, $y_2 := [y]_2 \cap [0]_1$, then $[\widetilde{0}(x_2)]_2 \cap [\widetilde{0}(x_3)]_3 =: x' \in [[\widetilde{0}(y_2)]_2 \cap [0]_3]_1$. This last statement is equivalent to (3). Consequently (1), (2) and (3) are equivalent.
(3) $\Leftrightarrow$ (4) Let $x \in \mathcal{P}$, $a := \big[[x]_3 \cap [0]_1\big]_2 \cap [0]_3$, $b := [x]_1 \cap [0]_3$ and $c := [x]_2 \cap [0]_3$. Then $c = a+b$, $-a = \big[[x]_3 \cap [0]_2\big]_1 \cap [0]_3$, $-b = \Big[\big[[b]_2 \cap [0]_1\big]_3 \cap [0]_2\Big]_1 \cap [0]_3$, $-c = \Big[\big[[x]_2 \cap [0]_1\big]_3 \cap [0]_2\Big]_1 \cap [0]_3$, and $-c = -a+(-b) \Leftrightarrow [-c]_2 \cap [-b]_1 \cap \big[[-a]_2 \cap [0]_1\big]_3 \neq \phi$. But $\widetilde{[0]}_2 \circ \widetilde{[0]}_1 \circ \widetilde{[0]}_3([x]_1) = \widetilde{[0]}_2 \circ \widetilde{[0]}_1([b]_2) = [-b]_1$, $\widetilde{[0]}_3 \circ \widetilde{[0]}_2 \circ \widetilde{[0]}_1([x]_2) = \widetilde{[0]}_3([-c]_1) = [-c]_2$ and $\widetilde{[0]}_1 \circ \widetilde{[0]}_3 \circ \widetilde{[0]}_2([x]_3) = \widetilde{[0]}_1 \circ \widetilde{[0]}_3([-a]_1) = \widetilde{[0]}_1([-a]_2) = \big[[-a]_2 \cap [0]_1\big]_3$. This shows the equivalence of (3) and (4). □

Remarks. 1. The statement (3) of (3.6) expresses that the bend-configuration $BE(0; \mathrm{id})$ of [5, Section 6] closes.
2. From (2.8) and (3.6) it follows that the point 0 is 2-extendable if and only if for $(E,+) := L(\mathcal{W};0;1,2)$ the map ν is an automorphism of $(E,+)$. If 0 is even

2-rotational then by (2.2), $\nu^2 = \text{id}$, and 0 is a characteristic 2 point if and only if $\nu = \text{id}$.
3. If a point $p \in \mathcal{P}$ is 2-rotational then by (2.8), $(\gamma_p)^3$ is uniquely extendable to an automorphism of $\mathcal{W}$ which we denote by $\widetilde{p}$ and which we call *reflection in the point p*; we have then $\widetilde{p}' = \text{id}(\in S_3)$, $p \in \text{Fix}(\widetilde{p})$, $\widetilde{p}^2 = \text{id}$, and $\widetilde{p} = \text{id} \Leftrightarrow$ p is a characteristic 2 point.
4. There are webs $\mathcal{W}$ with 2-rotational points p such that $\widetilde{p} \neq \text{id}$ and $|\text{Fix}(\widetilde{p})| \geq 2$.

Now we consider the map $0_6 = (132)_0 \in \Sigma_0$ and ask when $(0_6)_3 \in \text{Aut}(\mathcal{P}, \mathcal{G})$, i.e. $(0_6)_3(\mathcal{G}_3) = \mathcal{G}_2$ is true. For the answer we need the following (3.7):

3.7 *If (T, 0; i,j) is valid, then so is (T, 0; k, i).*

Proof. Let $x \in [0]_k$, $y \in [0]_i$ and $q := [0]_j \cap [[x]_j \cap [y]_k]_i$. Then $[y]_k \cap [q]_i = [x]_j \cap [y]_k$ shows $[0]_k \cap [[y]_k \cap [q]_i]_j = [0]_k \cap [[x]_j \cap [y]_k]_j = [0]_k \cap [x]_j = \{x\}$. By $(T, 0; i, j)$, $y \in [0]_i$ and $q \in [0]_j$ imply $\phi \neq [0]_k \cap [[y]_k \cap [q]_i]_j \cap [[q]_k \cap [y]_j]_i = \{x\} \cap [[q]_k \cap [y]_j]_i$, i.e. $x \in [[q]_k \cap [y]_j]_i$ and so $q \in [[x]_i \cap [y]_j]_k$. Hence the statement $(T, 0; k, i)$ is valid. □

3.8 Theorem. *The following statements are equivalent:*

(1) $\exists i \in \{1, 2, 3\} : (0_6)_i \in \text{Aut}(\mathcal{P}, \mathcal{G})$,
(2) $\forall i \in \{1, 2, 3\} : (0_6)_i \in \text{Aut}(\mathcal{P}, \mathcal{G})$,
(3) $(T, 0; 1, 2)$ *is satisfied,*
(4) $\forall a, b \in E : \sim b + (a + b) = a$, *i.e.,* $b + (a - b) = a$, *i.e.* $(E, +)$ *is a crossed-inverse loop ([1, Theorem 5.4]).*

Proof. Let $X \in \mathcal{G}_3 \setminus \{E\}$, $x := X \cap [0]_1$, $x_2 := X \cap [0]_2$. Then $0_6(x_2) = x$ and so if $(0_6)_3(\mathcal{G}_3) = \mathcal{G}_2$, then $(0_6)_3(X) = [x]_2$. Now let $p \in X$, $y := [p]_1 \cap [0]_2$, $p_1 := [p]_1 \cap [0]_3$, $p_2 := [p]_2 \cap [0]_3$, $q := [x]_2 \cap [y]_3$. Then firstly $x \in [0]_1$, $y \in [0]_2$, $p = [x]_3 \cap [y]_1$, $q = [y]_3 \cap [x]_2$ and $[0]_3 \cap [p]_2 \cap [q]_1 \neq \phi$ if $(T, 0; 1, 2)$ holds. Secondly $0_6(p_1) = y$, $0_6(p_2) = [p_2]_1 \cap [0]_2$, hence $(0_6)_3(p) = [0_6(p_1)]_3 \cap [0_6(p_2)]_1 = [y]_3 \cap [p_2]_1$. Consequently, $(0_6)_3(X) = [x]_2 \Leftrightarrow [x]_2 \cap [y]_3 \cap [p_2]_1 = \{q\} \cap [p_2]_1 \neq \phi \Leftrightarrow [0]_3 \cap [p]_2 \cap [q]_1 \neq \phi$. Hence $(0_6)_3(\mathcal{G}_3) = \mathcal{G}_2 \Leftrightarrow (T, 0; 1, 2)$. Now we set $a := [x]_2 \cap [0]_3$, $b := p_1$. Then $a + b = p_2$, $y = b\Box 0$, $[b\Box 0]_3 = [0\Box(\sim b)]_3$ and so $\sim b + (a + b) = a \Leftrightarrow [x]_2 \cap [y]_3 \cap [p_2]_1 \neq \phi$. With (3.7) all the statements are equivalent. □

Finally in our web we consider the orbits $[p]^i := \{\widetilde{X}(p) \mid X \in \mathcal{G}_i\}$ of a point $p \in \mathcal{P}$ with respect to the generators of $\mathcal{G}_i$, $i \in \{1, 2, 3\}$ and see by the definition that each orbit $[p]^i$ is an i-chain, hence $[p]^i \in \mathcal{C}_i$ and obtain the following theorem which is the case when i=3:

3.9 Theorem. *Let* $\mathcal{W} = (\mathcal{P}, \mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3)$ *be a web and let* $E \in \mathcal{G}_3$ *such that* $\widetilde{E} \in \text{Aut}(\mathcal{W})$. *Then*
(1) *If there is a chain* $D \in \mathcal{C}_3$ *such that* $\forall X \in \mathcal{G}_3 : \widetilde{D}(X) = X$ *(i.e.* $\mathcal{G}_3 \subset D^{\perp} \cup \{D\}$*), then* $\widetilde{D} \in \text{Aut}(\mathcal{W})$, $\widetilde{E} \circ \widetilde{D} = \widetilde{D} \circ \widetilde{E} \in \text{Aut}(\mathcal{W})^+$, *each point* $p \in E \cap D$ *is 2-rotational and* $\widetilde{p} = \widetilde{E} \circ \widetilde{D}$ *is the reflection in p and for each* $d \in D$, $D = [d]^3$,
(2) *If* $p \in E$ *is a 2-rotational point and* $\widetilde{p}$ *the reflection in p, then* $\widetilde{p} \circ \widetilde{E} = \widetilde{E} \circ \widetilde{p} \in \text{Aut}(\mathcal{W})$, $D := \text{Fix}(\widetilde{p} \circ \widetilde{E}) \in \mathcal{C}_3$, $\widetilde{D} = \widetilde{p} \circ \widetilde{E}$, $\mathcal{G}_3 \subset D^{\perp} \cup \{D\}$ *and* $D = [p]^3$.

Proof. (1) By (1.1.2), $\widetilde{D} \in \mathrm{Aut}(\mathcal{P}, \mathcal{G}_1 \cup \mathcal{G}_2)^-$ and by $\widetilde{D}(X) = X$ for $X \in \mathcal{G}_3$, hence $\widetilde{D} \in \mathrm{Aut}(\mathcal{W})$ and by (1.1.1) $\widetilde{X} = \widetilde{\widetilde{D}(X)} = \widetilde{D} \circ \widetilde{X} \circ \widetilde{D}$. Since $\widetilde{X}$ and $\widetilde{D}$ are involutions, $\widetilde{X} \circ \widetilde{D} = \widetilde{D} \circ \widetilde{X}$, in particular $\widetilde{E} \circ \widetilde{D} = \widetilde{D} \circ \widetilde{E}$. Since $\widetilde{E}, \widetilde{D} \in \mathrm{Aut}(\mathcal{W}) \cap \mathrm{Aut}(\mathcal{P}, \mathcal{G}_1 \cup \mathcal{G}_2)^-$, we obtain $\widetilde{E} \circ \widetilde{D} = \widetilde{D} \circ \widetilde{E} \in \mathrm{Aut}(\mathcal{W})^+$. Now let $p \in E \cap D, x \in E \setminus \{p\}, \gamma = (132), x' := (\gamma_p)^3(x), q := (\gamma_p)(x) = [x]_1 \cap [p]_2$, $q' := (\gamma_p)^2(x) = [p]_1 \cap [q]_3 = [p]_1 \cap [x']_2$. Then by $p \in D$, $\widetilde{D}([p]_2) = [p]_1$ and $\widetilde{D}([q]_3) = [q]_3$ since $[q]_3 \in \mathcal{G}_3$. Consequently $\widetilde{D}(q) = \widetilde{D}([p]_2 \cap [q]_3) = \widetilde{D}([p]_2) \cap \widetilde{D}([q]_3) = [p]_1 \cap [q]_3 = q'$ and so $\widetilde{D}(x) = \widetilde{D}([p]_3 \cap [q]_1) = [p]_3 \cap \widetilde{D}([q]_1) = [p]_3 \cap [\widetilde{D}(q)]_2 = [p]_3 \cap [q']_2 = x'$. Hence $\widetilde{E} \circ \widetilde{D}|_E = \widetilde{E} \circ \widetilde{D}|_{[p]_3} = (\gamma_p)^3|_E$ and so by the unique extendability, $\widetilde{E} \circ \widetilde{D} = \widetilde{p}$. Now for $d \in D$ and $X \in \mathcal{G}_3$, let $\widetilde{X}(d) \in [d]^3$, then $\widetilde{D} \circ \widetilde{X}(d) = \widetilde{X} \circ \widetilde{D}(d) = \widetilde{X}(d)$, hence $\widetilde{X}(d) \in \mathrm{Fix}(\widetilde{D}) = D$ by (1.1.1), i.e. $[d]^3 \subset D$. So we have $D = [d]^3$, since D and $[d]^3$ are both in C_3.
(2) By hypothesis, $\widetilde{p}$ and $\widetilde{E}$ are involutory automorphisms of $\mathcal{W}$ and $p \in E$, hence $\widetilde{E} \circ \widetilde{p} \circ \widetilde{E} = \widetilde{\widetilde{E}(p)} = \widetilde{p}$ and so $\widetilde{E} \circ \widetilde{p} = \widetilde{p} \circ \widetilde{E} \in \mathrm{Aut}(\mathcal{W})$ with $(\widetilde{E} \circ \widetilde{p})' = (1, 2)$, i.e. $\widetilde{E} \circ \widetilde{p} \in \mathrm{Aut}(\mathcal{P}, \mathcal{G}_1 \cup \mathcal{G}_2)^-$. By (1.1.4), $D := \mathrm{Fix}(\widetilde{E} \circ \widetilde{p}) \in \mathfrak{C}_3$ and $\widetilde{D} = \widetilde{E} \circ \widetilde{p}$. Finally let $X \in \mathcal{G}_3$. If $X = E$, then by $p \in E$, $\widetilde{D}(E) = \widetilde{E} \circ \widetilde{p}(E) = \widetilde{E}(E) = E$. Therefore let $X \neq E$ and let $q := [p]_2 \cap X$, $q' = [p]_1 \cap X$. Then $q' = \gamma_p(q)$, $\widetilde{p}(q) = (\gamma_p)^3(q)$ and $\widetilde{E}(q') = \widetilde{p}(q)$. Thus $\widetilde{E} \circ \widetilde{p}(X) = \widetilde{E} \circ \widetilde{p}([q]_3) = [\widetilde{E} \circ \widetilde{p}(q)]_3 = [q']_3 = X$, i.e. $\mathcal{G}_3 \subset D^\perp \cup \{D\}$ and by (1), $D = [p]^3$. □

Together with the results (4.2.3) and (6.4) of [5] we can state:

3.10 *For a web $\mathcal{W} = (\mathcal{P}, \mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3)$ let $\mathcal{P}_2$ be the set of all 2-extendable points. Then for $\mathcal{W}$ the following statements are equivalent:*

(1) $\mathcal{P}_2 \neq \phi$ *and* $\exists i \in \{1, 2, 3\} : \widetilde{\mathcal{G}_i} \subset \mathrm{Aut}(\mathcal{W})$ *(In this case, if* $0 \in \mathcal{P}_2$ *and* $j, k \in \{1, 2, 3\} \setminus \{i\}$ *with* $j \neq k$, *then* $D := [0]^i \in \mathfrak{C}_i$ *with* $D \subset \mathcal{P}_2$ *and* $\mathcal{G}_i \subset D^\perp \cup \{D\}$ *and* $(E, +) := L(\mathcal{W}; 0; j, k)$ *is a Bruck-loop),*
(2) $\exists 0 \in \mathcal{P}$ *and* $j, k \in \{1, 2, 3\}, j \neq k$ *such that* $(E, +) = L(\mathcal{W}; 0; j, k)$ *is a Bruck-loop (In this case* $0 \in \mathcal{P}_2$ *and* $\widetilde{\mathcal{G}_i} \subset \mathrm{Aut}(\mathcal{W})$*)),*
(3) $\exists i \in \{1, 2, 3\} : \widetilde{\mathcal{G}_i} \subset \mathrm{Aut}(\mathcal{W})$ *and* $\exists D \in \mathfrak{C}_i$ *with* $\mathcal{G}_i \subset D^\perp \cup \{D\}$.

Remark. If for a web $\mathcal{W} = (\mathcal{P}, \mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3)$ and an $i \in \{1, 2, 3\}$, $\widetilde{\mathcal{G}_i} \subset \mathrm{Aut}(\mathcal{W})$, then $\forall p, q \in \mathcal{P}_2$, $[p]^i \subset \mathcal{P}_2$ and either $[p]^i \cap [q]^i = \phi$ or $[p]^j = [q]^j$.

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References

[1] V.D. Belousov, Algebraic Nets and Quasigroups. *Acad. Sciences of M.S.S.R.* (1971).
[2] R.H. Bruck, *A Survey of Binary Systems.* Springer-Verlag, 1958.
[3] W. Blaschke and G. Bol, *Geometrie der Gewebe.* Springer-Verlag, 1938.
[4] M. Funk and P. T. Nagy, On collineation groups generated by Bol reflections *J. Geometry* **48** (1993), 63–78.
[5] E. Gabrieli, B. Im, and H. Karzel, Webs related to K-loops and reflection structures. *Abh. Math. Sem. Uni. Hamb.* **69** (1999), 89–102.

[6] H. KARZEL and H. J. KROLL, Perspectivities in circle geometries. *Geometry von Staudt's point of view.* Ed. by P. PLAUMANN and K. STRAMBACH. Dordrecht-Boston-London, 1981, p. 51–99.

[7] ———, *Geschichte der Geometrie seit Hilbert.* Wiss. Buchgesellschaft, 1988.

[8] H. KIECHLE, *Theory of K-loops.* Habilitationsschrift, Univ. Hamburg, 1998.

[9] G. KIST, Theorie der verallgemeinerten kinematischen Räume. *Beiträge zur Geometrie und Algebra* **14**, TUM-M 8611 (1986), 1–142.

[10] A. KREUZER, Inner mappings of Bruck loops. *Math. Proc. Camb.Phil.Soc.* **123** (1998), 53–57.

[11] K. REIDEMEISTER, Topolgische Fragen der Differentialgeometric. V. Gewebe und Gruppen. *Math. Z.* **29** (1929), 427–435.

[12] G. THOMSEN, Topologische Fragen der Differentialgeometric. XII. Schnittpunktsätze in ebenen Geweben *Abh. Math. Sem. Univ. Hamburg* **7** (1930), 99–106.

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