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Some K-Loops with Fixed Point Free Precession-Maps

Hubert Kiechle

Introduction

Let $(L, \oplus)$ be a loop, i.e., L is a set with a binary operation $\oplus$ such that for all $a, b \in L$ the equations $a \oplus x = b$ and $y \oplus a = b$ have unique solutions $x, y \in L$, and such that there exists $0 \in L$ with $a \oplus 0 = 0 \oplus a = a$. The condition $a \oplus (b \oplus x) = (a \oplus b) \oplus \delta_{a,b}(x)$ for $a, b, x \in L$ then clearly defines a bijective map $\delta_{a,b} : L \to L$. Following [13] we use the phrase *precession-maps* to denote these maps.

L is called a *K-loop* if the precession-maps are automorphisms of the loop, if the automorphic inverse property is satisfied, and the Bol identity holds. To be precise, we require that for all $a, b, c, d \in L$

$$\begin{aligned} \delta_{a,b}(c \oplus d) &= \delta_{a,b}(c) \oplus \delta_{a,b}(d) \\ a \oplus (b \oplus (a \oplus c)) &= (a \oplus (b \oplus a)) \oplus c \quad \text{(Bol identity)}, \\ \ominus(a \oplus b) &= (\ominus a) \oplus (\ominus b) \qquad \text{(automorphic inverse property)} \end{aligned}$$

where $\ominus a$ is defined by $a \oplus (\ominus a) = 0$. The Bol identity implies the equality of left and right inverse, i.e., $(\ominus a) \oplus a = 0$ as well. Thus we use the term "automorphic inverse property" in the standard way. It should be noted that in the definition of K-loops the Bol identity is usually replaced by conditions on the precession-maps. Using [7] one easily deduces that the two definitions are equivalent.

The notion of a K-loop evolved from the study of neardomains $(F, \oplus, \cdot)$, where in particular $(F, \oplus)$ is a K-loop with fixed point free precession-maps. Since every automorphism fixes 0, the phrase "fixed point free" refers to the situation, where only the identity has fixed points other that 0. Presently, there seem to be no examples of proper neardomains known. "Proper" here means "not a nearfield". A nearfield is a neardomain where all precession-maps are $\mathbf{1}$. See [14; V §1] for definitions and more information concerning nearfields and neardomains, and their connection with sharply 2-transitive groups.

The set of admissible velocities $\mathbf{R}^3_c := \{v \in \mathbf{R}^3; |v| < c\}$ together with the relativistic velocity addition forms a K-loop. This has been proved by Ungar (cf. [12, 13]). Encouraged by Ungar's discovery, many K-loops have been constructed in recent years (cf. [1, 2, 4, 5, 6, 7, 8, 9]). In case of Ungar's example $\mathbf{R}^3_c$, the precession-maps are the "Thomas-precessions" or "Thomas-rotations" of special relativity, hence the name.

Here we give a variation of a construction due to Karzel, which evolved from his attempt to give an "elegant representation of [Ungar's] results" (cf. [4; p. 339]). While Karzel used positive definite, hermitian 2×2-matrices over euclidian fields, we will consider symmetric matrices of determinant 1 over pythagorean fields.

In a forthcoming paper, we will give a generalization of our construction, to include Karzel/Ungar's examples, and Im's work [1, 2]. This will involve $n \times n$-matrices over pythagorean fields. There we will try to present conceptual arguments. Here we confine ourselves with direct calculations.

Basic to both papers is the work of Kreuzer and Wefelscheid [9]. They describe a very general, yet powerful method to construct K-loops from groups in the following way: Given a group G with a subgroup Ω and a set L of representatives for the cosets of Ω in G. This means in particular $G = L\Omega$. For $A, B \in L$ let $A \oplus B$ be the unique element in L such that $AB \in (A \oplus B)\Omega$. Define $d_{A,B} \in \Omega$ by $AB = (A \oplus B)d_{A,B}$. Assume that the following conditions are satisfied for all $g \in \Omega$ and for all $A, B \in L$:

$$\mathbf{1} \in L, \quad gLg^{-1} \subseteq L, \quad ABd_{A,B}^{-1} = d_{A,B}AB, \quad ALA \subseteq L,$$

then $(L, \oplus)$ is a K-loop. Note that $\mathbf{1}$ becomes the 0-element of the loop. The precession-maps are given by $\delta_{A,B}(X) = d_{A,B}Xd_{A,B}^{-1}$. This is the content of [9; (3.8),(3.7)].

The most striking feature of the examples presented here is the fact that all non-identity precession-maps are fixed point free. To my knowledge there is only one other example with this property in the literature [5]. Examples of this kind might open the road to a proper neardomain.

According to [9; (7.2)] one can construct a Frobenius group from a K-loop with fixed point free precession-maps. This generalizes the construction of sharply 2-transitive groups from neardomains. We will not go into this further and refer the reader to the literature.

1. The Construction

Let R be an ordered, pythagorean field and let L be the set of symmetric, positive definit 2×2-matrices of determinant 1. Write $\Omega := \mathrm{SO}(2, R)$. To be precise

$$L = \left\{ \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} \in R^{2\times 2}; \alpha > 0, \alpha\gamma - \beta^2 = 1 \right\},$$

$$\Omega = \left\{ \begin{pmatrix} u & -v \\ v & u \end{pmatrix} \in R^{2\times 2}; u^2 + v^2 = 1 \right\}.$$

For two elements $A = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix}, B = \begin{pmatrix} \alpha' & \beta' \\ \beta' & \gamma' \end{pmatrix}$ from L we let

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} := AB = \begin{pmatrix} \alpha\alpha' + \beta\beta' & \alpha\beta' + \gamma'\beta \\ \alpha'\beta + \gamma\beta' & \gamma\gamma' + \beta\beta' \end{pmatrix}$$

and

$$\Delta^2 := (a + d)^2 + (b - c)^2 = a^2 + b^2 + c^2 + d^2 + 2.$$

The last equality follows from $\det AB = ad - bc = 1$. Since R is pythagorean, $\Delta \in R$. Hence we can define

$$A \oplus B := \frac{1}{\Delta}\begin{pmatrix} a^2+b^2+1 & ac+bd \\ ac+bd & c^2+d^2+1 \end{pmatrix}$$

and

$$d_{A,B} := \frac{1}{\Delta}\begin{pmatrix} a+d & b-c \\ c-b & a+d \end{pmatrix} \in \Omega.$$

We show first

(1.1) $A \oplus B = ABd_{A,B}^{-1} \in L$.

Proof. We have to prove that $\Delta^2 AB = \Delta^2(A \oplus B)d_{A,B} =$

$$\begin{pmatrix} (a^2+b^2+1)(a+d)+(ac+bd)(c-b) & (a^2+b^2+1)(b-c)+(ac+bd)(a+d) \\ (ac+bd)(a+d)+(c^2+d^2+1)(c-b) & (ac+bd)(b-c)+(c^2+d^2+1)(a+d) \end{pmatrix}.$$

Using $ad - bc = 1$, we can compute $(a^2+b^2+1)(a+d)+(ac+bd)(c-b) = a(a^2+b^2+1+ad+c^2-bc)+d(b^2+1+bc-b^2) = a(a^2+b^2+c^2+d^2+2) = a\Delta^2$, hence the (1,1)-entry behaves as claimed. In a very similar way, one can verify three more equalities for the other three entries to obtain the first assertion.

This first assertion implies $\det A \oplus B = 1$. By construction, $A \oplus B$ is symmetric and positive definit, hence it is an element of L. ■

We record the main claim in a

Theorem. *$(L, \oplus)$ is a K-loop with fixed point free precession-maps. More precisely, Ω acts on L by conjugation as a fixed point free automorphism group, which contains all precession-maps.*

2. The Proof

We will show a series of lemmas, aiming at the hypothesis of [9; (3.8)], described in the introduction.

As a preparation, we prove that any $A \in L$ can be diagonalized over R.

(2.1) *The eigenvalues of* $A = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} \in L$ *are*

$$\lambda := \frac{1}{2}(\alpha+\gamma+\sqrt{(\alpha-\gamma)^2+4\beta^2}) \quad \text{and} \quad \lambda^{-1}.$$

These are elements of R. There exits $D \in \Omega$ such that $A = D\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} D^{-1}$.

Proof. The roots of the characteristic polynomial are

$$\frac{1}{2}\left(\alpha+\gamma \pm \sqrt{(\alpha+\gamma)^2-4(\alpha\gamma-\beta^2)}\right) = \frac{1}{2}\left(\alpha+\gamma \pm \sqrt{(\alpha-\gamma)^2+(2\beta)^2}\right) \in R,$$

since the radiant is a sum of squares. The "+" sign yields λ. Since $\det A = 1$, the other eigenvalue is λ^{-1}. An eigenvector (u, v) can be normalized in R. The other eigenvector is necessarily orthogonal and can be normalized as well. If the matrix with these eigenvectors as columns happens to have negative determinant, simply replace one eigenvector by its negative to obtain D. ■

Remark. In fact, the ground field is pythagorean if and only if every symmetric, positive definit 2×2-matrix is diagonalizable, cf. [11; Lemma 1]. Thus there seems to be no hope to enlarge the class of ground fields without new methods.

The preceding lemma allows us in quite a few cases to assume without loss of generality that one element from L is diagonalized. This simplifies some calculations considerably. We shall use this lemma in the sequel without specific reference.

(2.2) $E := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in L$, *$L^{-1} = L$ and $gLg^{-1} = L$ for all $g \in \Omega$.*

Proof. The first two assertions are obvious. For the third, note that conjugating a symmetric matrix by an orthogonal matrix yields a symmetric matrix with the same eigenvalues, hence the result. ■

(2.3) *Let $A, B \in L$, then the cosets $A\Omega$ and $B\Omega$ are equal if and only if $A = B$.*

Proof. Let $B^{-1}A \in \Omega$. Looking at $gB^{-1}Ag^{-1} = (gBg^{-1})^{-1}gAg^{-1}$ for appropriate $g \in \Omega$, we can assume without loss of generality by (2.1) that

$$A = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}.$$

So $B^{-1}A = \begin{pmatrix} \lambda^{-1}\alpha & \lambda^{-1}\beta \\ \lambda\beta & \lambda\gamma \end{pmatrix} \in \Omega$ implies $\lambda^{-1}\beta = -\lambda\beta$, hence $\beta(\lambda + \lambda^{-1}) = 0$. Now, $\lambda + \lambda^{-1} = 0$ would imply $\lambda^2 + 1 = 0$, contradicting $\lambda \in R$. Thus $\beta = 0$ and $B^{-1}A \in \Omega$ is diagonal, hence equal $\pm E$. Since $\lambda > 0$ and $\alpha > 0$, we conclude $\lambda^{-1}\alpha = 1$ and $\lambda\gamma = 1$, which implies $A = B$. ■

The above lemmas entail

(2.4) *$L\Omega$ is a subgroup of $\mathrm{SL}(2, R)$.*

Proof. Let $A, B \in L$ and $g, h \in \Omega$. Using (2.2) and (1.1), we find $AgBh = AgBg^{-1}gh = AgBg^{-1}d_{A,gBg^{-1}}^{-1}d_{A,gBg^{-1}}gh \in L\Omega$, and using (2.2) again $(Ag)^{-1} = g^{-1}A^{-1}gg^{-1} \in L\Omega$. ■

Remark. Employing polar decomposition of matrices [10; p. 155/56], one can show that in fact $L\Omega = \mathrm{SL}(2, R)$.

(2.5) *For all $A, B \in L$ and $g \in \Omega$ the condition $ABg \in L$ implies $g = d_{A,B}^{-1}$. Furthermore, $ABd_{A,B}^{-1} = d_{A,B}BA$.*

Proof. $ABg \in L$ puts AB in the uniquely determined coset $ABg\Omega$. But AB is also in $ABd_{A,B}^{-1}\Omega$ by. From (1.1) and (2.3) we conclude $ABg = ABd_{A,B}^{-1}$, which in turn implies $g = d_{A,B}^{-1}$.

From (1.1) follows $ABd_{A,B}^{-1} = (ABd_{A,B}^{-1})^{\mathrm{T}} = d_{A,B}BA$. Here we use $g^{\mathrm{T}} = g^{-1}$ for all $g \in \Omega$. ■

(2.6) *$ABA \in L$ for all $A, B \in L$.*

Proof. Assuming $A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ and $B = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix}$, we find $ABA = \begin{pmatrix} \lambda^2\alpha & \beta \\ \beta & \lambda^{-2}\gamma \end{pmatrix}$, which is clearly in L. ■

By (2.4), (2.3), (2.2), (2.5) and (2.6) the hypothesis of [9; (3.8)] are fulfilled, hence $(L, \oplus)$ is a K-loop. Before we complete the proof of our theorem we will show

(2.7) *For all $g \in \Omega$ the map $\widehat{g} : L \to L, X \mapsto gXg^{-1}$ is an automorphism of $(L, \oplus)$. Moreover, the following conditions are equivalent.*

(I) *$\widehat{g}$ has a fixed point $\neq E$;*

(II) *$\widehat{g} = \mathbf{1}$;*

(III) *$g = \pm E$.*

Proof. We have $\widehat{g}(A \oplus B)\widehat{g}(d_{A,B}) = \widehat{g}(AB) = \widehat{g}(A)\widehat{g}(B) = (\widehat{g}(A) \oplus \widehat{g}(B))d_{\widehat{g}(A),\widehat{g}(B)}$. By uniqueness of the decomposition (2.3) we must have $\widehat{g}(A \oplus B) = \widehat{g}(A) \oplus \widehat{g}(B)$.

The implications "(III) $\Longrightarrow$ (II) $\Longrightarrow$ (I)" are obvious.

To show "(I) $\Longrightarrow$ (III)", we consider

$$C := \begin{pmatrix} u & -v \\ v & u \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \begin{pmatrix} u & v \\ -v & u \end{pmatrix} = \begin{pmatrix} \lambda u^2 + \lambda^{-1}v^2 & (\lambda - \lambda^{-1})uv \\ (\lambda - \lambda^{-1})uv & \lambda v^2 + \lambda^{-1}u^2 \end{pmatrix}$$

with $\lambda \neq 1, \lambda > 0$ and $g = \begin{pmatrix} u & -v \\ v & u \end{pmatrix} \in \Omega$, i.e., $u^2 + v^2 = 1$. In particular, $\lambda \neq \lambda^{-1}$. The condition $C = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ implies $(\lambda - \lambda^{-1})uv = 0$, hence $uv = 0$.

The case $u = 0$ leads to $v = \pm 1$. It follows that $C = \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix}$, a contradiction. Thus $v = 0$ and $u = \pm 1$, as was to be shown. ■

So our theorem will be proved if we can show

(2.8) *For all $A, B \in L$, the map $\delta_{A,B} : L \to L, X \mapsto d_{A,B}Xd_{A,B}^{-1}$ is a fixed point free automorphism of $(L, \oplus)$, with the property $A \oplus (B \oplus X) = (A \oplus B) \oplus \delta_{A,B}(X)$.*

Proof. The $\delta_{A,B}$'s are the precession-maps by [9; (3.7)]. We have just shown that they are fixed point free automorphisms. ■

We remark that our construction can be carried over to $\mathrm{GL}(2, R)$ by choosing all symmetric, positive definite matrices for L and $\mathrm{O}(2, R)$ for Ω. However, the precession-maps are not fixed point free anymore, since there are diagonal matrices different from E in L. Our future paper mentioned in the introduction provides a better frame to discuss this matter.

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