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Results in Mathematics

Polar Graphs and Corresponding Involution Sets, Loops and Steiner Triple Systems

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Abstract. A 1-factorization (or parallelism) of the complete graph with loops $(P, \mathcal{E}, \|)$ is called *polar* if each 1-factor (parallel class) contains exactly one loop and for any three distinct vertices x_1, x_2, x_3, if $\{x_1\}$ and $\{x_2, x_3\}$ belong to a 1-factor then the same holds for any permutation of the set $\{1, 2, 3\}$. To a polar graph $(P, \mathcal{E}, \|)$ there corresponds a polar involution set $(P, \mathcal{I})$, an idempotent totally symmetric quasigroup $(P, *)$, a commutative, weak inverse property loop $(P, +)$ of exponent 3 and a Steiner triple system $(P, \mathcal{B})$.

We have: $(P, \mathcal{E}, \|)$ satisfies the trapezium axiom $\Leftrightarrow \forall \alpha \in \mathcal{I} : \alpha\mathcal{I}\alpha = \mathcal{I} \Leftrightarrow (P, *)$ is self-distributive $\Leftrightarrow (P, +)$ is a Moufang loop $\Leftrightarrow (P, \mathcal{B})$ is an affine triple system; and: $(P, \mathcal{E}, \|)$ satisfies the quadrangle axiom $\Leftrightarrow \mathcal{I}^3 = \mathcal{I} \Leftrightarrow (P, +)$ is a group $\Leftrightarrow (P, \mathcal{B})$ is an affine space.

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Introduction

This note is part of our investigations on regular involution sets $(P, \mathcal{I})$, complete graphs with parallelism $(P, \mathcal{E}, \|)$ and involutorial difference loops $(P, +)$.

Any regular involution set $(P, \mathcal{I})$, by the choice of a fixed point $o \in P$, gives rise to a *regular reflection structure* $(P, {}^\circ; o)$ in the sense of [3] and conversely a regular reflection structure is nothing else than a regular set of involutions acting on a set P where a point o has been fixed.

From $(P, {}^\circ; o)$ both a quasigroup "$\circ$" and a loop operation "$+$" can be derived by setting $a \circ b := a^\circ(b)$ and $a + b := a^\circ o^\circ(b) = (a \circ o) \circ (o \circ b)$ (where a° is the unique involution of the set $\mathcal{I}$ mapping o to a). The quasigroup $(P, \circ)$ has o as right identity and involutorial left multiplications (*semisymmetric quasigroup*,

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i.e., $\forall a, b \in P : a \circ (a \circ b) = b$), while the loop $(P, +)$, with identity o, is a principal loop isotope of $(P, \circ)$ (in the sense of [1]) and it is characterized by the property that left differences are involutions (*involutorial difference loop*, i.e., $\forall a, b \in P : a - (a - b) = b$).

Start now from any semisymmetric quasigroup $(P, \bullet)$ and pick an element o, then define an isotopic semisymmetric quasigroup $(P, \circ)$ with right identity o by $a \circ b := \varphi_o(a) \bullet b$ where $\varphi_o(x)$ is the uniquely determined element of P such that $\varphi_o(x) \bullet o = x$, for all $x \in P$. So the left multiplications of $(P, \circ)$ make up a regular reflection structure with respect to o. By different choices of the point o, the corresponding quasigroups and loops change in the same isotopism class.

On the other side, a regular involution set can be represented in a natural way by means of a 1-factorization (or parallelism) of the complete graph (with loops) on the vertex set P, each 1-factor representing a given involution of $\mathcal{I}$.

So regular involution sets, semisymmetric quasigroups and 1-factorizations of complete graphs with loops are coextensive and any of these structures, by different choices of a point o, gives rise to an isotopism class of involutorial difference loops and of semisymmetric quasigroups with right identity o.

Assuming that such a loop $(P, +)$ is a *Bol loop* (a *K-loop*, or *Bruck-loop*, in fact), the corresponding quasigroup $(P, \circ)$ is a *Bol quasigroup* as well, the involution set is *invariant* and the 1-factorization of the graph satisfies the so-called *trapezium axiom* (cf. §2 and [8]).

Finally, when $(P, +)$ is a group, hence an abelian group, the quasigroup is characterized by the property $a \circ (b \circ (c \circ x)) = c \circ (b \circ (a \circ x))$ for all $a, b, c, x \in P$, the involution set is *3-reflectional* (i.e., $\mathcal{I}^3 = \mathcal{I}$) and the 1-factorization of the corresponding graph satisfies the so-called *quadrangle axiom* (see §2).

Consider now for the involution set the further assumption $|\text{Fix } \alpha| = 1$ for each $\alpha \in \mathcal{I}$, say $\{a\} := \text{Fix } \alpha$ and write $\tilde{a} := \alpha$ (in this case in any point o the reflection structure $(P, ^\circ; o)$ is *with midpoints*, in the sense of [4]). Thus a new quasigroup operation can be defined by setting for $a, b \in P$, $a \cdot b := \tilde{a}(b)$, so that $(P, \cdot)$ is *idempotent and semisymmetric* and the involutorial difference loop $(P, +)$ derived as before in each point $o \in P$ is also a principal isotope of $(P, \cdot)$, now characterized by the further property that for all $a \in P$ the equation $a - x = x$ admits a unique solution. Hence, the 1-factorization of the corresponding graph, since each 1-factor contains exactly one loop, reduces to a *near 1-factorization* (in the sense of [10]) of the complete simple graph (i.e., without loops) on the vertex set P.

In this case the Bol condition implies that $(P, +)$ is a *strongly 2-divisible* K-loop (i.e., the map $x \to 2x$ is a permutation of P), the reflection structure is *invariant with midpoints*, the idempotent, semisymmetric quasigroup $(P, \cdot)$ is also *left-distributive* (i.e., $a \cdot bc = ab \cdot ac$ $\forall a, b, c \in P$) and the near 1-factorization of the complete graph on P fulfils the trapezium axiom.

Finally we consider **polar structures**, i.e., for $(P, \mathcal{I})$ we make the assumptions:

1. $\forall \alpha \in \mathcal{I}, \quad |\text{Fix } \alpha| = 1$

2. for $a, b \in P$, denoting by $\widetilde{ab} \in \mathcal{I}$ be the involution of $\mathcal{I}$ interchanging a and b and by $a * b = \mathrm{Fix}\ \widetilde{ab}$, then $a * (a * b) = b$.

In this case we obtain (cf. (3.3.i)) that $(P, *)$ is exactly the idempotent and semi-symmetric quasigroup $(P, \cdot)$ which turns out to be also commutative, hence totally symmetric. The corresponding loop $(P, +)$ is then commutative of exponent 3 (cf. (3.3)) and the corresponding graph is polar (cf. §3).

Moreover, in this case, we have an extension of these correspondences: if

$$\mathcal{B} := \left\{ \{a, b, a * b\} \mid \{a, b\} \in \binom{P}{2} \right\} {}^{1}$$

then $(P, \mathcal{B})$ is a Steiner triple system and conversely, to any Steiner triple system corresponds a polar involution set (cf. (3.4)).

Making the further assumption that $\mathcal{I}$ is invariant, then the commutative loop $(P, +)$ is a K-loop hence a commutative Moufang loop of exponent 3, the polar graph $(P, \mathcal{E}, \|)$ satisfies the trapezium axiom (cf. (2.1)) and the Steiner triple system $(P, \mathcal{B})$ is an affine triple system (i.e., a pseudo-affine space in the sense of [6, 13]) (cf. (5.1)) which becomes an affine space if and only if $(P, +)$ is a group (cf. (5.2)).

1. Notations and known results

Let P be a non empty set, we denote, as usual, by $\mathrm{Sym}P$ the whole permutation group of P and let $J := \{\sigma \in \mathrm{Sym}P \mid \sigma^2 = id\}$, $J^* := J \setminus \{id\}$.

For a fixed subset $\mathcal{I} \subseteq J$, the pair $(P, \mathcal{I})$ is called an **involution set**.

An involution set $(P, \mathcal{I})$ is called

- **regular** if $\forall a, b \in P: \ \exists_1 \gamma \in \mathcal{I}$ with $\gamma(a) = b$
- **invariant** if $\forall \xi \in \mathcal{I}: \ \xi \mathcal{I} \xi = \mathcal{I}$.

We recall now some definitions concerning quasigroups, loops and graphs.

If P is provided with a binary operation "+" such that:

1. $\forall a, b \in P \quad \exists_1 (x, y) \in P \times P$ such that $a + x = b = y + a$
   then $(P, +)$ is called a **quasigroup**.
   A quasigroup $(P, +)$ is called:
   - **semisymmetric** if $a + (a + b) = b$, for all $a, b \in P$,
   - **idempotent** if $a + a = a$, for all $a \in P$,
   - **left-distributive** if $a + (b + c) = (a + b) + (a + c)$, for all $a, b, c \in P$.

If a quasigroup $(P, +)$ satisfies also the condition:

2. $\exists\ 0 \in P$ such that $\forall a \in P: \ 0 + a = a + 0 = a$
   then $(P, +)$ is called a **loop** (with identity 0).

[1]For $n \in \mathbf{N}$, let $\binom{P}{n}$ denote the set of all subsets of P consisting of exactly n elements.

An **isotopism** from the quasigroup $(P, +)$ to the quasigroup $(P', +')$ is a triple (α, β, γ) of bijective maps from P to P' such that $\forall a, b \in P : \alpha(a) +' \beta(b) = \gamma(a+b)$ (cf. [1]). In the case $\alpha = \beta = \gamma$ the notion reduces to that of isomorphism.

In a quasigroup $(P, +)$, for any $a \in P$ let $a^+ : P \to P; x \to a + x$ and $P^+ := \{a^+ \mid a \in P\}$ then P^+ is a regular subset of $\mathrm{Sym}P$, containing the identity if and only if $(P, +)$ is a loop.

In a loop $(P, +)$, consider the **negative map** $\nu : P \to P; x \to -x$, where for any $a \in P$ the element $-a$ is the *right inverse* of a defined by $a + (-a) = 0$. Note that ν is an involution if and only if $-a + a = a + (-a) = 0$ for all $a \in P$ (i.e., $(P, +)$ is a *loop with inverses*).

A loop $(P, +)$ is called an **involutorial difference loop** (cf. [5]) if

$(\star)$ $\forall a, b \in P : \ a - (a - b) = b.$

We shall denote such a loop briefly by **$(\star)$-loop**.

A loop $(P, +)$ is called:

- a **Bol loop** if $\forall a, b \in P : \ a^+ b^+ a^+ = (a + (b + a))^+$,
- a **K-loop** (or Bruck loop, cf. [9]) if it is a Bol loop and moreover $\nu \in \mathrm{Aut}(P, +)$, i.e., $\forall a, b \in P : -(a + b) = -a + (-b)$ (*automorphic inverse property*),
- a **Moufang loop** if: $\forall a, b \in P \quad a^+ b^+ a^+ = ((a + b) + a)^+$.

We note that any $(\star)$-Bol loop is a K-loop and that a commutative loop is a K-loop if and only if it is Moufang.

For further notions on quasigroups and loops we shall refer to [1] and [14].

Let $\Gamma := (P, \mathcal{E})$ be a complete graph in which P denotes the set of vertices and $\mathcal{E} := \binom{P}{1} \cup \binom{P}{2}$ the set of edges[2]. An equivalence relation "$\|$" defined on $\mathcal{E}$ is called a **parallelism** if the following *parallel axiom* is satisfied

(**P**) $\forall p \in P, \forall D \in \mathcal{E}, \quad \exists_1 E \in \mathcal{E}$ with $p \in E$ and $D \parallel E$.

Note that a parallelism on the graph $(P, \mathcal{E})$ is exactly a **1-factorization** in the sense of [10], each parallel class corresponding to a *1-factor*.

In the sequel, we shall denote by $(P, \mathcal{E}, \|)$ a complete graph with parallelism.

From the definitions one can deduce the following correspondences among the previous structures (see f.i. [7, 8]):

(1.1) *Let $(P, +)$ be a $(\star)$-loop and let $\mathcal{I} := P^+\nu$, then $(P, \mathcal{I})$ is a regular involution set.*

(1.2) *Let $(P, \mathcal{I})$ be a regular involution set, for $a, b, c, d \in P$ let $\widetilde{ab} \in \mathcal{I}$ be the unique element of $\mathcal{I}$ such that $\widetilde{ab}(a) = b$ (if $a = b$ we shall write simply $\tilde{a}$ instead of $\widetilde{aa}$) and define on $\mathcal{E} := \binom{P}{1} \cup \binom{P}{2}$, $\{a, b\} \parallel \{c, d\} \Leftrightarrow \widetilde{ab} = \widetilde{cd}$, then $(P, \mathcal{E}, \|)$ is a complete graph with parallelism.*

[2] We observe that the notion of a complete graph here corresponds to what usually in the literature is called (*complete*) *graph with loops*.

(**1.3**) *Let* $(P, \mathcal{E}, \|)$ *be a complete graph with parallelism, for* $a, b \in P$ *let* $\widetilde{ab} : P \to P; x \to x'$ *where* x' *is defined by* $\{a, b\} \parallel \{x, x'\}$ *and let* $\mathcal{I} := \{\widetilde{ab} \mid a, b \in P\}$. *Then* $(P, \mathcal{I})$ *is a regular involution set.*

(**1.4**) *Let* $(P, \mathcal{I})$ *be a regular involution set; if* $o \in P$ *is a fixed element then:*
   i) $(P, \circ)$ *where "*$\circ$*" is defined by* $a \circ b := \widetilde{oa}(b)$ *is a semisymmetric quasigroup with right identity* o,
   ii) $(P, +_o)$ *where "*$+_o$*" is defined by* $a +_o b := \widetilde{oa}\tilde{o}(b)$ *is a* $(\star)$*-loop with identity* o,
   iii) $(P, +_o)$ *is a principal isotope of* $(P, \circ)$, *in fact* $a +_o b = (a \circ o) \circ (o \circ b)$.

Starting from a regular involution set $(P, \mathcal{I})$ and fixing any element $o \in P$, we shall call the binary operation "$+_o$" the $\mathbf{K_o}$**-derivation** and we shall write $+_o = K_o(P, \mathcal{I})$. In the $(\star)$-loop $(P, +_o)$, let ν_o be the corresponding negative map.

It is straightforward to verify that by different choices of the point o, the corresponding quasigroups (as in (1.4.i)) are isotopic hence the derived loops are isotopic as well. We show directly such an isotopism:

(**1.5**) *Let* $(P, +_o)$ *and* $(P, +_{o'})$ *be the loops obtained from a regular involution set* $(P, \mathcal{I})$ *by the K-derivation in two distinct elements* $o, o' \in P$. *Then* $(\Theta, \tilde{o}'\tilde{o}, id)$ *with* $\Theta : P \to P; x \to \widetilde{ox}(o')$, *is an isotopism from* $(P, +_o)$ *to* $(P, +_{o'})$.

If $\mathcal{I}$ *is also invariant then the K-derivations in two distinct points are isomorphic.*

Proof. First we remark that for all $x \in P$, $\widetilde{o'\Theta(x)} = \widetilde{ox}$ since $\mathcal{I}$ is regular, hence the equation $\Theta(x) = a$ has the unique solution $x = \widetilde{o'a}(o)$. This implies that the map Θ is a permutation of P. Thus: $\Theta(x) +_{o'} (\tilde{o}'\tilde{o}(y)) = \widetilde{o'\Theta(x)}\tilde{o}'\tilde{o}'\tilde{o}(y) = \widetilde{ox}\tilde{o}(y) = x +_o y$.

If $\mathcal{I}$ is invariant the involution $\widetilde{oo'}$ is an isomorphism between $(P, +_o)$ and $(P, +_{o'})$ (see e.g., [7]). $\square$

(**1.6**) *Let* $(P, +)$ *be a* $(\star)$*-loop and for* $a, b, c, d \in P$ *let* x, *resp.* y, *be the solution of* $x - a = b$, *resp.* $y - c = d$. *Let* $\{a, b\} \parallel \{c, d\} \Leftrightarrow x = y$, *then*

$$\left(P, \mathcal{E} := \binom{P}{1} \cup \binom{P}{2}, \|\right)$$

is a complete graph with parallelism.

Proof. We have only to prove that "$\|$" is an equivalence relation on the set $\mathcal{E}$ fulfilling the condition (P). By $(\star)$, $x - a = b$ implies $x - b = a$ so $\{a, b\} \parallel \{b, a\}$ and "$\|$" is reflexive; the symmetric and transitive properties are trivial. Let $\{a, b\} \in \mathcal{E}$ and $c \in P$; if e denotes the unique solution of $x - a = b$ then $\{a, b\} \parallel \{c, e - c\}$ so (P) is valid. $\square$

In Section 3 we shall relate the previous concepts to that of Steiner triple system which is defined as follows:

Let P be a non empty set of *points* and let $\emptyset \neq \mathcal{B} \subseteq 2^P$ (the elements of $\mathcal{B}$ are called *blocks*), the pair $(P, \mathcal{B})$ is called a **Steiner triple system** if:

i) $\forall p, q \in P, p \neq q \quad \exists_1 B \in \mathcal{B}$ such that $p, q \in B$,
ii) $\forall B \in \mathcal{B} \quad |B| = 3$.

2. Characteristic configurations

In this section let $(P, \mathcal{E}, \parallel)$ be a complete graph with parallelism, $\mathcal{I}$ the corresponding regular involution set and for $o \in P$ let $+_o := K_o(P, \mathcal{I})$ be the K_o-derivation and ν_o the negative map of the $(\star)$-loop $(P, +_o)$. By [8] we have:

(**2.1**) *The following statements are equivalent:*
- i) *the regular involution set* $\mathcal{I}$ *is invariant;*
- ii) $\exists\ o \in P:\ (P, +_o)$ *is a Bol loop;*
- iii) $\exists\ o \in P:\ (P, +_o)$ *is a K-loop;*
- iv) $\forall\ o \in P:\ (P, +_o)$ *is a K-loop;*
- v) $(P, \mathcal{E}, \parallel)$ *satisfies the* **trapezium axiom**, *i.e.,*
  (**T**) $\forall (a, b, c, d), (a', b', c', d') \in P^4$:

$$\{a, b\} \parallel \{a', b'\} \parallel \{c, d\} \parallel \{c', d'\}, \{b, c\} \parallel \{b', c'\} \Rightarrow \{a, d\} \parallel \{a', d'\}.$$

(**2.2**) *The following statements are equivalent:*
- i) $\exists\ o \in P:\ (P, +_o)$ *is a group (and then a commutative group);*
- ii) $\mathcal{I}^3 = \mathcal{I}$;
- iii) $(P, \mathcal{E}, \parallel)$ *satisfies the* **quadrangle axiom**, *i.e.,*
  (**Q**) $\forall (a, b, c, d), (a', b', c', d') \in P^4$:

$$\{a, b\} \parallel \{a', b'\}, \{b, c\} \parallel \{b', c'\}, \{c, d\} \parallel \{c', d'\} \Rightarrow \{a, d\} \parallel \{a', d'\}.$$

Proof.

"i) $\Rightarrow$ ii)" If $(P, +_o = +)$ is a group then $(\star)$ can be written in the form: $x = a-(a-x) = a+x-a$, i.e., $(P, +)$ is a commutative group and therefore for $a^+\nu, b^+\nu, c^+\nu \in P^+\nu = \mathcal{I}$ we have: $(a^+\nu)(b^+\nu)(c^+\nu)(x) = a-(b-(c-x)) = (a-b+c)^+\nu(x)$ and so $(a^+\nu)(b^+\nu)(c^+\nu) \in \mathcal{I}$ hence $\mathcal{I}^3 = \mathcal{I}$.

"ii) $\Rightarrow$ i)" Now let $\mathcal{I}^3 = \mathcal{I}$, then for all $a, b \in P$, $a^+b^+ = \widetilde{oa}\tilde{o}\widetilde{ob}\tilde{o}$ and $\widetilde{oa}\tilde{o}\widetilde{ob}(o) = a + b$. Hence, by (ii) and the regularity of $\mathcal{I}$, $\widetilde{oa}\tilde{o}\widetilde{ob} = \widetilde{o(a+b)}$ and thus $a^+b^+ = \widetilde{o(a+b)}\tilde{o} = (a+b)^+$.

"ii) $\Rightarrow$ iii)" Let a, b, c, d and a', b', c', d' be two quadrangles with $\{a, b\} \parallel \{a', b'\}$, $\{b, c\} \parallel \{b', c'\}$, $\{c, d\} \parallel \{c', d'\}$ i.e., $\widetilde{ab} = \widetilde{a'b'}$, $\widetilde{bc} = \widetilde{b'c'}$, $\widetilde{cd} = \widetilde{c'd'}$. Since $\widetilde{ab}\widetilde{bc}\widetilde{cd}(d) = a$ we have that ii) implies $\widetilde{ab}\widetilde{bc}\widetilde{cd} = \widetilde{ad} = \widetilde{a'b'}\widetilde{b'c'}\widetilde{c'd'} = \widetilde{a'd'}$ and so $\{a, d\} \parallel \{a', d'\}$.

"iii) $\Rightarrow$ ii)" Let $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{I}$ and for any $x \in P$ let $x_1 = \alpha_1(x)$, $x_2 = \alpha_2(x_1)$, $x_3 = \alpha_3(x_2)$, then $\alpha_3\alpha_2\alpha_1(x) = \widetilde{xx_3}(x)$ and, by the regularity of $\mathcal{I}$, $\alpha_1 = \widetilde{xx_1} = \widetilde{yy_1}$, $\alpha_2 = \widetilde{x_1x_2} = \widetilde{y_1y_2}$, $\alpha_3 = \widetilde{x_2x_3} = \widetilde{y_2y_3}$ for $x, y \in P$, hence $\{x, x_1\} \parallel \{y, y_1\}$, $\{x_1, x_2\} \parallel \{y_1, y_2\}$, $\{x_2, x_3\} \parallel \{y_2, y_3\}$ and so, by iii), $\{x, x_3\} \parallel \{y, y_3\}$ i.e., $\widetilde{xx_3} = \widetilde{yy_3}$. This shows $\alpha_3\alpha_2\alpha_1 = \widetilde{xx_3} \in \mathcal{I}$. $\square$

3. Graphs with polar triangles

Let $(P, \mathcal{E}, \|)$ be a complete graph with parallelism, let $(P, \mathcal{I})$ with $\mathcal{I} := \{\widetilde{ab} \mid a, b \in P\}$, be the corresponding regular involution set and let $(P, +)$ with $+ := K_o(P, \mathcal{I})$, for $o \in P$, be the corresponding loop. From now on we assume that

(F) $\forall \alpha \in \mathcal{I}, \quad |\mathrm{Fix}\ \alpha| = 1$

hence, according to the notation introduced in (1.2), if $\alpha(a) = a$ then $\alpha = \tilde{a}$, so $\mathcal{I} = \{\tilde{a} \mid a \in P\} =: \tilde{P}$. Note that in the finite case this assumption is fulfilled only if $|P|$ is odd.

Now, defining $a \cdot b := \tilde{a}(b)$, $(P, \cdot)$ is an idempotent semisymmetric quasigroup such that $a + b = a' \cdot (o \cdot b)$ (where a' is defined by $a' \cdot o = a$), so $(P, +)$ is a principal isotope of $(P, \cdot)$.

In the graph, the condition corresponding to (F) is:

(F′) for any $A \in \mathcal{E}$ the parallel class $A_\| := \{E \in \mathcal{E} \mid E \parallel A\}$ of A contains exactly one edge X with $|X| = 1$.

In this situation, considering the complete simple graph $\left(P, \binom{P}{2}\right)$, the parallelism determines a **near 1-factorization** of $\left(P, \binom{P}{2}\right)$, that is a partition of the edge set $\binom{P}{2}$ into *near 1-factors,* each covering all verices but one (see [10]).

Finally, in the loop $(P, +)$ the corresponding condition is:

(F″) $\forall a \in P\ \exists_1 x \in P : \ a - x = x$.

In this case, it turns out that $(P, +)$ is a Bol loop if and only if the reflection structure is *invariant with midpoints*, what is equivalent to saying that the idempotent, semisymmetric quasigroup $(P, \cdot)$ is also left-distributive. Hence (see e.g., [12], Propositions 9.8 and 9.11) the loop $(P, +)$ is a *strongly 2-divisible* K-loop, i.e., the map $x \to 2x$ is a permutation of P.

For the complete (simple) graph on P it happens that the near 1-factorization induced by the involution set $(P, \mathcal{I})$ fulfils the trapezium axiom.

From our assumptions we have:

(3.1) *For $a, b \in P$ let $a * b :=$ Fix $\widetilde{ab}$. Then $(P, *)$ is a commutative idempotent quasigroup.*

Proof. By definition and (F), "$*$" is a commutative and idempotent bynary operation on P. Now $b = a * x =$ Fix $\widetilde{ax}$ implies $\tilde{b} = \widetilde{ax}$ hence $x = \tilde{b}(a)$ is the unique solution of the equation $a * x = b$. □

A triple $\{a, b, c\} \in \binom{P}{3}$ is called a **triangle in a** if $\{a\} \parallel \{b, c\}$ ($\Leftrightarrow \tilde{a} = \widetilde{bc} \Leftrightarrow a = b * c$).

Any two distinct vertices $\{a, b\}$ determine exactly one triangle, namely the triangle $\{a * b, a, b\}$ in $a * b$. A triangle is called **semipolar**, resp. **polar**, if {a,b,c} is a triangle in two, resp. all vertices.

(**3.2**) *For all* $\{a, b\} \in \binom{P}{2}$ *the triangle* $\{a * b, a, b\}$ *is*

$$\begin{array}{rccc} semipolar\ if & a * (a * b) = b & or & b * (b * a) = a\,, \\ polar\ if & a * (a * b) = b & and & b * (b * a) = a\,. \end{array}$$

The graph $(P, \mathcal{E}, \|)$ with (F$'$) is called a **polar graph** if all triangles are polar or equivalently if

$$\forall a, b \in P: \ a * (a * b) = b.$$

Correspondently, $(P, \mathcal{I} = \tilde{P})$ is called a **polar involution set** if it satisfies (F) and

(π) $\quad \forall a, b \in P: \ \tilde{a}(b) = \tilde{b}(a).$

(**3.3**) *Let* $(P, \mathcal{I})$ *be a polar involution set, let* $o \in P$ *be fixed and* $+ := K_o(P, \mathcal{I})$. *Then:*
  i) $\forall a, b \in P: \ a * b = \tilde{a}(b) = a \cdot b$ and $\widetilde{ab} = \widetilde{a * b} = \widetilde{\tilde{a}(b)}$
  ii) $(P, +)$ *is a commutative* $(\star)$*-loop such that:*
  $a + b = (o * a) * (o * b), \ \ o * a = -a \ \ and \ a * b = -a - b.$

Proof. i) Let $c :=$ Fix $\widetilde{ab} = a * b$ then by (π), $\widetilde{cb} = \tilde{a}$ hence $\tilde{a}(b) = \widetilde{cb}(b) = c = a * b$ and by the regularity of $(P, \mathcal{I})$, we have $\widetilde{ab} = \tilde{c} = \widetilde{a * b}$.

ii) By i) follows: $(o{*}a){*}(o{*}b) = \widetilde{o * a}(o{*}b) = \widetilde{oa}(\tilde{o}(b)) = a{+}b$, $o{*}a = \tilde{o}(a) = -a$ and $\widetilde{o(-a)} = \widetilde{\tilde{o}(-a)} = \tilde{a}$ implying $-a - b = \widetilde{o(-a)}\tilde{o}(-b) = \tilde{a}(b) = a * b$. $\square$

Remark. Note that in a commutative loop $(P, +)$ the property $(\star)$ is equivalent to the so-called *weak inverse property* (cf. [11, 14]).

In $(P, \mathcal{E}, \|)$ we denote by $\mathcal{B}$ the set of all polar triangles of the graph.

(**3.4**) *Let* $(P, \mathcal{E}, \|)$ *be a complete graph with parallelism satisfying* (F'), *let* $(P, \mathcal{I})$ *be the corresponding regular involution set with* (F), *let* $o \in P$ *be fixed,* $+ := K_o(P, \mathcal{I})$ *and* $(P, +)$ *be the corresponding* $(\star)$*-loop with* (F''). *Then the following statements are equivalent:*
  i) $(P, \mathcal{E}, \|)$ *is a polar graph,*
  ii) $(P, \mathcal{I})$ *is a polar involution set,*
  iii) $(P, \mathcal{B})$ *is a Steiner triple System*
  (*hence if* $|P| = n \in \mathbf{N}$ *then* $n \equiv 1, 3$ *mod 6*),
  iv) $(P, +)$ *is a commutative loop of exponent 3*
  (*i.e.,* $\forall x \in P: \ x + (x + x) = \ o$).

Proof.

"i) $\Leftrightarrow$ ii) $\Leftrightarrow$ iii)" follows from the definitions.

"iii) $\Rightarrow$ iv)" The set of all triangles in o is given by $\{\{o, a, \tilde{o}(a) = -a\} \mid a \in P \setminus \{o\}\}$. For $a \in P \setminus \{o\}$, let $2a$ be the solution of $x - a = a$, then $\tilde{a} = (2a)^+\nu$ and the triangle $\{o, a, -a\}$ is polar if $\tilde{a}(o) = (2a)^+\nu(o) = 2a = -a$ and $2(-a) = a$ or equivalently if

$$1) \quad -a - a = a \quad \text{and} \quad 2) \quad a + a = -a.$$

From 2) we obtain $a + (a + a) = a + (-a) = o$.

Finally let $\{a, b\} \in \binom{P}{2}$ then $\{a, \tilde{b}(a), b\}$ is a triangle in b and, by assumption, a polar triangle hence also a triangle in a implying $-a - b = (2a)^+\nu(b) = \tilde{a}(b) = \tilde{b}(a) = -b - a$, i.e., $(P, +)$ is commutative.

"iv) $\Rightarrow$ i)" Let $(P, +)$ be commutative of exponent 3 and let $\{a, b\} \in \binom{P}{2}$. Then $o = a + (a + a) = \widetilde{oa}\tilde{o}\widetilde{oa}\tilde{o}(a)$ implies $a = \tilde{o}\widetilde{oa}\tilde{o}(a) = \widetilde{o(-a)}(a)$ hence $\tilde{a} = \widetilde{o(-a)}$ and so $-a - b = \widetilde{o(-a)}\tilde{o}(-b) = \tilde{a}(b) = -b - a = \tilde{b}(a)$. Moreover $\widetilde{\tilde{a}(b)}(a) = -a - \tilde{a}(b) = -a - (-a - b) = b$ and this tells us that $\{a, b, \tilde{a}(b)\}$ is a polar triangle. $\square$

Remark. If $(P, +)$ is any commutative $(\star)$-loop of exponent 3 and if we set $\forall a \in P : \tilde{a} := (a + a)^+\nu$ then $(P, \tilde{P} := \{\tilde{a} \mid a \in P\})$ is a polar involution set.

In fact (F'') is satisfied because for all $a \in P \quad a - (-a) = a + a = -a$ so $-a$ is a solution of the equation $a - x = x$. Moreover, for any $x \in P$ such that $a - x = x$ we have, by the commutativity: $a - x = x \Rightarrow -x + (a - x) = -x + x = 0 \Rightarrow -x + a = -(-x) = -x - x \Rightarrow a = -x$ so $x = -a$ is the unique solution. Hence $(P, \tilde{P})$ satisfies (F) and, by (3.4), it is polar.

(3.5) *Let $(P, \mathcal{B})$ be a Steiner triple system, for $\{a, b\} \in \binom{P}{2}$ let $\overline{a, b}$ be the uniquely determined block containing a and b. Define:*

$$\tilde{a} : \begin{cases} P \to P \\ x \to \overline{a, x} \setminus \{a, x\} & \text{if } x \neq a \\ x \to a & \text{if } x = a \end{cases}$$

and let $\tilde{P} := \{\tilde{a} \mid a \in P\}$. Then $(P, \tilde{P})$ is a polar involution set and the corresponding graph $(P, \mathcal{E}, \|)$ is polar.

4. Substructures and automorphisms

In this section let again $(P, \mathcal{I})$ be a polar involution set. Since P is provided with the structures $(P, \mathcal{I})$, $(P, *)$, $(P, \mathcal{B})$, we have the following three possibilities ensuring that a non empty set $S \subseteq P$ is a substructure:

1. $\forall a, b \in S : \widetilde{ab}(S) = S$;
2. $\forall a, b \in S : a * b \in S$;
3. $\forall \{a, b\} \in \binom{S}{2} : \overline{a, b} \subseteq S$.

Since, for $a \neq b$, $\overline{a, b} = \{a, b, a * b\}$, the conditions 2. and 3. are equivalent. Now let S be substructure in the sense of 2., hence if $a, b \in S$ then $c := a * b \in S$ and if $x \in S$ then $\widetilde{ab}(x) = \tilde{c}(x) = c * x \in S$. Conversely, if S is a substructure according to 1. for $a, b \in S \quad c = a * b = \tilde{a}(b) \in S$.

Therefore we call a subset $S \subseteq P$ a **subinvolution set** if one of the equivalent conditions 1., 2., 3. is satisfied. In this case if $\tilde{S} := \{\tilde{s} \mid s \in S\}$ then $(S, \tilde{S})$ is again a polar involution set and $(S, *)$ is a subquasigroup of $(P, *)$.

Let $\mathcal{S}$ be the set of all substructures of $(P, \mathcal{I})$. Then $\mathcal{B} \subseteq \mathcal{S}$ and the automorphism groups:

$$\begin{aligned} \operatorname{Aut}(P, \mathcal{I}) &:= \{\sigma \in \operatorname{Sym} P \mid \forall a, b \in P : \widetilde{\sigma(a)\sigma(b)} = \sigma\widetilde{ab}\sigma^{-1}\}, \\ \operatorname{Aut}(P, *) &:= \{\sigma \in \operatorname{Sym} P \mid \forall a, b \in P : \sigma(a * b) = \sigma(a) * \sigma(b)\}, \\ \operatorname{Aut}(P, \mathcal{B}) &:= \{\sigma \in \operatorname{Sym} P \mid \forall B \in \mathcal{B} : \sigma(B) \in \mathcal{B}\} \end{aligned}$$

coincide.

We observe that:

1. If $\sigma \in \operatorname{Sym} P$ then: $\sigma \in \operatorname{Aut}(P, \mathcal{I}) \Leftrightarrow \sigma\mathcal{I}\sigma^{-1} = \mathcal{I} \Leftrightarrow \forall a, b \in P : \sigma(a) * \sigma(b) = \sigma(a * b)$.
2. If $a \in P$, $S \in \mathcal{S}(a) := \{S \in \mathcal{S} \mid a \in S\}$ then $\tilde{a}(S) = S$.

(**4.1**) *For $a \in P$ we have: $\tilde{a} \in \operatorname{Aut}(P, \mathcal{B}) \Leftrightarrow \forall b, c \in P : a * (b * c) = (a * b) * (a * c)$.*

Proof. If $B = \{x, y, x * y\} \in \mathcal{B}$ then $\tilde{a}(B) = \{\tilde{a}(x), \tilde{a}(y), \tilde{a}(x * y)\} \in \mathcal{B} \Leftrightarrow \tilde{a}(x * y) = \tilde{a}(x) * \tilde{a}(y)$ and so $a * (x * y) = (a * x) * (a * y)$. $\square$

(**4.2**) *Let $o \in P$ be fixed, let $+ := K_o(P, \mathcal{I})$ and let $S \subseteq P$ then S is a subloop of $(P, +) \Leftrightarrow o \in S \in \mathcal{S}$. In particular $\forall a \in P \setminus \{o\} : \{o, a, o * a\}$ is a subloop of $(P, +)$.*

5. Invariant polar involution sets and corresponding graphs and loops

In this section we make the further assumption that the polar involution set $(P, \mathcal{I})$ is invariant. Hence, by (2.1), the K-derivation "+" in any point gives rise to a K-loop and the graph satisfies the trapezium axiom (T). Consider now the associated Steiner triple system according to (3.4). Recall that, considering the blocks as lines, a Steiner triple system $(P, \mathcal{B})$ is a linear space where each line is incident with exactly three points and $\mathcal{S}$ is the set of all subspaces. We recall that the subspaces generated by three non collinear points are called **planes** and that a linear space is called a **pseudo-affine space** (cf. [6, 13]) if each plane is an affine plane. Moreover a pseudo-affine space where there exists a line with four points is already an affine space (cf. [2, 6]).Therefore every proper pseudo-affine space is a Steiner triple system which is called **affine triple system** (cf. [14]).

The following theorem clarifies the relations existing among all these structures.

(**5.1**) *Let* $(P, \mathcal{I})$ *be a polar involution set,* $(P, \mathcal{E}, \|)$ (*resp.* $(P, *)$, *resp.* $(P, \mathcal{B})$) *be the corresponding polar graph* (*resp. idempotent, totally symmetric quasigroup, resp. Steiner triple system*) *and let* $o \in P$ *be fixed and* $(P, +)$ *with* $+ = K_o(P, \mathcal{I})$ *be the corresponding commutative loop of exponent 3. Then the following conditions are equivalent:*

i) $(P, \mathcal{I})$ *is invariant,*
ii) $\widetilde{P} := \{\tilde{a} \mid a \in P\} = \mathcal{I} \subseteq \mathrm{Aut}(P, \mathcal{I})$,
iii) $P^* := \{a^* \mid a \in P\} \subseteq \mathrm{Aut}(P, *)$, *i.e.,* $(P, *)$ *is left-distributive,*
iv) $(P, +)$ *is a commutative K-loop and so a commutative Moufang loop of exponent 3,*
v) $(P, \mathcal{E} \|)$ *is a polar graph with* (T),
vi) $(P, \mathcal{B})$ *is a Steiner triple system with* $P^* \subseteq \mathrm{Aut}(P, \mathcal{B})$,
vii) $(P, \mathcal{B})$ *is an affine triple system* (*i.e., a pseudo-affine space of order 3*).

Proof. Since, by definition, $\mathrm{Aut}(P, \mathcal{I})$ is the normalizer of $\mathcal{I}$ in $\mathrm{Sym}P$ the statements i) and ii) are equivalent and since $\mathrm{Aut}(P, \mathcal{I}) = \mathrm{Aut}(P, \mathcal{B})$, "ii) $\Leftrightarrow$ iii) $\Leftrightarrow$ iv) $\Leftrightarrow$ vi)" follow from (4.1) and (3.3). By (2.1) and (3.4) "i) $\Leftrightarrow$ iv) $\Leftrightarrow$ v)". The equivalence of vi) and vii) is a consequence of a Theorem of M. Hall (cf. e.g., [14]). □

Moreover, for the group case we have:

(**5.2**) *Under the assumption of* (*5.1*), *the following conditions are equivalent:*

i) $\mathcal{I}^3 = \mathcal{I}$,
ii) $(P, +)$ *is an abelian group,*
iii) $(P, \mathcal{E}, \|)$ *is a polar graph with* (Q),
iv) $(P, \mathcal{B})$ *is an affine space.*

Proof. By (2.2) we have the equivalence of i), ii) and iii). The equivalence of ii) and iv) is proved in [14], theor. (17). □

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