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Results in Mathematics

Legendre-like Theorems in a General Absolute Geometry

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Abstract. In this paper the axiomatic basis will be a *general absolute plane* $\mathbf{A} = (\mathcal{P}, \mathcal{L}, \alpha, \equiv)$ in the sense of [6], where $\mathcal{P}$ and $\mathcal{L}$ denote respectively the set of *points* and the set of *lines*, α the *order structure* and $\equiv$ the *congruence*, and where furthermore the word "general" means that no claim is made on any kind of continuity assumptions. Starting from the classification of general absolute geometries introduced in [5] by means of the notion of congruence, singular or hyperbolic or elliptic, we get now a complete characterization of the different possibilities which can occur in a general absolute plane studying the value of the angle δ defined in any Lambert–Saccheri quadrangle or, equivalently, the sum of the angles of any triangle. This yelds, in particular, a Archimedes-free proof of a statement generalizing the classical "*first Legendre theorem*" for absolute planes.

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1. Introduction

In the classical absolute geometry, hence under the assumption of Hilbert's axioms including the continuity or the Archimedes' axiom (cf., [2], p. 154 or [6], p. 104), if one considers the sum of the angles of a triangle (for an explicit definition of these notions see the following Sections 2, 4), denoting by R the right angle, the following result is well known:

(1.1) *In an absolute plane* $\mathbf{A}$ *fulfilling the continuity and the Archimedes' axiom, let* σ *be the sum of the angles of a triangle then:*

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1. *$\sigma = 2R$ if and only if* **A** *is the Euclidean plane, i.e., the parallel axiom is valid.*
2. *$\sigma < 2R$ if and only if* **A** *is the hyperbolic plane, i.e., the parallel axiom is not valid.*

So the value of the sum of the angles of any triangle characterizes the two types of continuous and Archimedean absolute geometries. Such a characterization can be pursued in the absolute plane also by means of another peculiar configuration, the so called *Lambert–Saccheri quadrangle*, i.e., a quadrangle (a, b, c, d) with $\overline{d, a} \perp \overline{a, b} \perp \overline{b, c} \perp \overline{c, d}$. If we denote by $\delta := \angle(c, d, a)$ then the following result is true in the classical absolute plane (see e.g., [1], § 9.6 and [2], p. 166):

(1.2) *In an absolute plane* **A** *fulfilling the continuity and the Archimedes' axiom, let (a, b, c, d) be a Lambert–Saccheri quadrangle with $\delta := \angle(c, d, a)$ and σ be the sum of the angles of any triangle, then:*

1. *$\delta = R$ if and only if* **A** *is the Euclidean plane and so $\sigma = 2R$.*
2. *$\delta < R$ if and only if* **A** *is the hyperbolic plane and so $\sigma < 2R$.*

In this paper the axiomatic basis will be, as in the previous papers [3], [4] and [5], a *general absolute plane* $\mathbf{A} = (\mathcal{P}, \mathcal{L}, \alpha, \equiv)$ in the sense of [6], where $\mathcal{P}$ and $\mathcal{L}$ denote respectively the set of *points* and the set of *lines*, α the *order structure* and $\equiv$ the *congruence*, and where furthermore the word "general" means that no claim is made on any kind of continuity assumptions.

Note that within this frame, the hyperbolic geometry is not anymore the unique alternative to the Euclidean one when the Euclid's parallel axiom is removed, but it has to be characterized by the validity of the *hyperbolic parallel axiom* **H** (see Section 5).

Given any Lambert–Saccheri quadrangle (a, b, c, d) in the general absolute plane **A**, denote by $a' := (a \perp \overline{c, d}) \cap \overline{c, d}$; then the set LS of all Lambert–Saccheri quadrangles splits into the three classes: the *rectangles* LS_r, the *hyperbolic* LS_h and the *elliptic quadrangles* LS_e characterized respectively by $a' = d$, by $a' \in\,]c, d[$ and by $d \in\,]a', c[$.

In [5] the wellknown statement (the so called *"second Legendre theorem"*) "$LS_r \neq \emptyset$ implies $LS = LS_r$" (cf., e.g., [6], (21.3)) was supplemented by "$LS_h \neq \emptyset$ implies $LS = LS_h$" and "$LS_e \neq \emptyset$ implies $LS = LS_e$". In the first case the absolute plane **A** is called *singular*, otherwise **A** is an *ordinary* absolute plane ([2], p. 162). Therefore, the congruence $\equiv$ of **A** was defined there *singular* (or *Euclidean*), *hyperbolic* or *elliptic* if, respectively, $LS_r \neq \emptyset$, $LS_h \neq \emptyset$, or $LS_e \neq \emptyset$.

Starting from the classification of general absolute geometries introduced in [5] by means of the notion of congruence, singular or hyperbolic or elliptic, we get now a complete characterization of the different possibilities which can occur in a general absolute plane studying the value of the angle δ defined in any Lambert–Saccheri quadrangle or, equivalently, the sum of the angles of a triangle.

First of all, in order to have a clear and complete formal basis for the following rational deductions, in Section 2 we give a definition of sum and of an order relation for angles, valid in a general (not necessarily continuous nor Archimedean) absolute geometry with the aim to give sense also to the case that the sum of angles could be greater than $2R$ (the *straight angle*). By means of these tools, in Sections 3, 4, 5 we prove the following results which, besides supplementing the classification of general absolute planes (Theorem 1), also entail a Archimedes-free proof of a statement (Theorem 2) extending the classical "*first Legendre theorem*"[1] to any general absolute plane with singular or hyperbolic congruence. Actually we will prove the following theorems:

1. (see Theorem 1, Section 3) *In any Lambert–Saccheri quadrangle* (a, b, c, d)*, denoting by* δ *the angle* $\angle(c, d, a)$*, it holds:*

$$\delta = R \textit{ if and only if the congruence} \equiv \textit{is singular ;}$$

$$\delta < R \textit{ if and only if the congruence} \equiv \textit{is hyperbolic ;}$$

$$\delta > R \textit{ if and only if the congruence} \equiv \textit{is elliptic .}$$

2. (see Theorem 2, Section 4) *If* σ *is the sum of the angles of any triangle then:*

$$\sigma = 2R \textit{ if and only if the congruence} \equiv \textit{ is singular};$$

$$\sigma < 2R \textit{ if and only if the congruence} \equiv \textit{ is hyperbolic};$$

$$\sigma > 2R \textit{ if and only if the congruence} \equiv \textit{ is elliptic}.$$

3. (see Theorem 3, Section 5) *The congruence of a general hyperbolic plane is hyperbolic.*

In particular, Theorem 3, together with the second statements of Theorems 1 and 2, yields a proof for general hyperbolic planes just of the sufficient condition of the classical statements (**1.1**) (**1.2**), provided that the notion of hyperbolic plane is characterized among general absolute planes by the axiom **H** (see Section 5).

2. Basic notions, order and sum of angles

In this paper we shall consider an absolute plane in the sense of Hilbert, where however *we make no claim concerning continuity or the Archimedes' axiom.*

So, let $\mathbf{A} = (\mathcal{P}, \mathcal{L}, \alpha, \equiv)$ be an *absolute plane*. For this notion and the basic definitions and properties of *order* α and *congruence* $\equiv$ we refer to [6], § 16. Here we just recall some notation.

The order structure α is defined through the so called *order function* $(L \mid a, b) \in \{-1, +1\}$, for $L \in \mathcal{L}$ and $a, b \in \mathcal{P} \setminus L$ as in [6], § 13.

Given any two distinct points $a, b \in \mathcal{P}$ and any line $G \in \mathcal{L}$, denote by $\overline{a, b} \in \mathcal{L}$ the uniquely determined line through a and b, by $]a, b[:= \{x \in \overline{a, b} \mid (x \mid a, b) = -1\}$ the *open segment* of extremes a and b, by $(a \perp G)$ the uniquely determined line through a and *perpendicular* to G.

[1] See, e.g., M. Dehn, *Die Legendre'schen Sätze über die Winkelsumme im Dreieck.* Math. Ann. Vol. 53 (1900), 404–439.

Denote by $\tilde{a} : P \to P;\ x \mapsto x' := \tilde{a}(x)$ such that a is the midpoint of (x, x') for $x \neq a$ and $x' = a$ if $x = a$, the *reflection in the point* a.

Furthermore, let $\widetilde{G}$ denote the *reflection in the line* G, i.e., the permutation of $\mathcal{P}$ which maps any point $x \in \mathcal{P}$ to the point $x' := \widetilde{G}(x)$ such that G is the midline of (x, x') if $x \notin G$, and $x' = x$ if and only if $x \in G$.

Both point- and line-reflections are indeed *motions* of the absolute plane, namely permutations of the point set $\mathcal{P}$ that preserve the line structure $\mathcal{L}$, the order α and the congruence $\equiv$ of $\mathbf{A}$. Set $\widetilde{\mathcal{P}} := \{\tilde{a} \mid a \in \mathcal{P}\}$ and $\widetilde{\mathcal{L}} := \{\widetilde{G} \mid G \in \mathcal{L}\}$.

A set $\mathfrak{b}$ of lines is called *pencil* (= *Büschel* in [6], § 19) if

$$\bigcup \mathfrak{b} = \mathcal{P}\ ,\ \forall A, B, C \in \mathfrak{b}\ :\ \widetilde{A} \circ \widetilde{B} \circ \widetilde{C} \in \widetilde{\mathcal{L}} \ \text{ and } \ \forall A, B \in \mathfrak{b} \quad \text{with} \quad A \neq B\ :$$

$$\mathfrak{b} = \left\{X \in \mathcal{L} \mid \widetilde{A} \circ \widetilde{B} \circ \widetilde{X} \in \widetilde{\mathcal{L}}\right\}.$$

By an *angle* β we understand an ordered pair of halflines $(\overrightarrow{b,a}; \overrightarrow{b,c})$ having the initial point b in common and we shall denote it by $\beta := \angle(\overrightarrow{b,a}; \overrightarrow{b,c}) = \angle(a, b, c)$. If a, b, c are not collinear β will be called *proper*.

If $a' := (a \perp \overline{b,c}) \ \cap\ \overline{b,c}$ and $c' := (c \perp \overline{b,a}) \ \cap\ \overline{b,a}$, we call β *acute* or *right* or *obtuse* depending whether $b \neq a'$ and $(b \mid c, a') = 1$ or $b = a' = c'$ or $b \neq a'$ and $(b \mid c, a') = -1$ and we write respectively $\beta < R$ or $\beta = R$ or $\beta > R$.

Let β be proper: the set of points $\hat{\beta} := \{x \in \mathcal{P} \mid (\overline{b,a} \mid c, x) = (\overline{b,c} \mid a, x) = 1\}$ is called the *interior* of β. The complement $\check{\beta} := \mathcal{P} \setminus (\hat{\beta} \cup \overrightarrow{b,a} \cup \overrightarrow{b,c})$ is called the *exterior* of β. We call the pair $(\beta, \hat{\beta})$ the *interior angle* and $(\beta, \check{\beta})$ the *exterior angle*.

If $2R$ denotes the *straight angle*, for any proper angle β we shall write:

$$(\beta, \hat{\beta}) < 2R\,, \quad (\beta, \check{\beta}) > 2R\,.$$

Two angles α and β are called *congruent* if there is a motion μ such that $\mu(\alpha) = \beta$ and *properly congruent* if μ is a proper motion.

If α, β are congruent we shall write $\alpha \equiv \beta$.

If $\beta := \angle(a, b, c)$ and $\beta' := \angle(a', b', c')$ are two proper angles such that $\overrightarrow{b,c} = \overrightarrow{b',c'}$ and either $a' \in \hat{\beta}$, or $\widetilde{\overline{b,c}}(a') \in \hat{\beta}$ then we shall write $\beta' < \beta$.

In the general case $\overrightarrow{b,c} \neq \overrightarrow{b',c'}$, let μ be the proper motion (which is uniquely determined, see [6], (17.15)) with $\mu(\overrightarrow{b',c'}) = \overrightarrow{b,c}$ then we define $\beta' < \beta$ if $\mu(\beta') < \beta$.

Proposition 2.1. *Let $\beta := \angle(a, b, c)$, $\beta' := \angle(a', b', c')$, $\beta'' := \angle(a'', b'', c'')$ be distinct proper angles such that $\overrightarrow{b,c} = \overrightarrow{b',c'} = \overrightarrow{b'',c''}$ and $\beta > \beta'$, $\beta' > \beta''$. Then we have $\beta > \beta''$.*

Proof. We may assume w.l.o.g. $(\overline{b,c} \mid a, a') = (\overline{b,c} \mid a, a'') = 1$ hence also $(\overline{b,c} \mid a', a'') = 1$. The assumption $\beta > \beta'$ implies $(\overline{b,a} \mid a', c) = (\overline{b,c} \mid a', a) = 1$, hence by [6], (13.12.2), we have $(\overline{b,a'} \mid a, c) = -1$. Thus by Axiom **(V1)** (cf., [6], § 16):

i) there is a point $m' := \overline{b,a'} \cap]a,c[$, so $(a \mid c,m') = 1$ and, since β is proper, $b \neq m'$ and $(b \mid a',m') = (\overline{b,a} \mid a',m') = (\overline{b,a} \mid a',c) \cdot (\overline{b,a} \mid c,m') = 1 \cdot (a \mid c,m') = 1$ which implies $\overrightarrow{b,a'} = \overrightarrow{b,m'}$. By $\beta' > \beta''$ we have also $-1 = (\overline{b,a''} \mid c,a') = (\overline{b,a''} \mid c,m') \cdot (\overline{b,a''} \mid m',a') = (\overline{b,a''} \mid c,m') \cdot (b \mid m',a') = (\overline{b,a''} \mid c,m')$. Thus:

ii) there exists an $m'' := \overline{b,a''} \cap]m',c[$ and we have $(b \mid a'',m'') = (\overline{b,a'} \mid a'',m'') = (\overline{b,a'} \mid a'',c) \cdot (\overline{b,a'} \mid c,m'') = 1 \cdot (m' \mid c,m'') = 1$ which implies $\overrightarrow{b,a''} = \overrightarrow{b,m''}$ and also $(m'' \mid m',c) = -1$.

iii) By i) and ii) we get $(m' \mid a,m'') = (m' \mid a,c) \cdot (m' \mid c,m'') = -1 \cdot 1 = -1$ hence $(m'' \mid a,m') = 1$, thus:

iv) $(m'' \mid a,c) = (m'' \mid a,m') \cdot (m'' \mid m',c) = 1 \cdot (-1) = -1$ hence $(a \mid m'',c) = 1$. Finally we get $(\overline{b,a} \mid a'',c) = (\overline{b,a} \mid a'',m'') \cdot (\overline{b,a} \mid m'',c) = (b \mid a'',m'') \cdot (a \mid m'',c) = 1 \cdot 1 = 1$. Hence $\beta > \beta''$. □

Proposition 2.2. *For any two distinct proper angles $\beta := \angle(a,b,c)$, $\beta' := \angle(a',b',c')$ with $\overrightarrow{b,c} = \overrightarrow{b',c'}$ it holds:*

$$\beta' < \beta \textit{ if and only if, either } \hat{\beta}' \cup \overrightarrow{b,a'} \subset \hat{\beta} \textit{ or } \widetilde{\overline{b,c}}(\hat{\beta}' \cup \overrightarrow{b,a'}) \subset \hat{\beta}\,.$$

Proof. First suppose $a' \in \hat{\beta}$: this implies directly $\overrightarrow{b,a'} \subset \hat{\beta}$. Furthermore, if we take any $x \in \hat{\beta}'$ then, for $\xi := \angle(x,b,c)$, by definition $\xi < \beta'$ and so by (**2.1**) $\xi < \beta$. Thus by definition $x \in \hat{\beta}$. The converse is obvious.

To complete the proof it is enough to observe that $\widetilde{\overline{b,c}}$ is a motion of the absolute plane **A** and that motions preserve the order function and then the order relation for angles. □

Let now $\beta := \angle(a,b,c)$ and $\beta' := \angle(a',b',c')$ such that $\overrightarrow{b,c} = \overrightarrow{b',a'}$ and $(\overline{b,c} \mid a,c') = -1$. Then $\overrightarrow{b,a}$, $\overrightarrow{b,c}$, $\overrightarrow{b,c'}$ are three distinct halflines and we define the following *sums* :

$\beta + \beta' := \angle(a,b,c') =: \sigma$
$(\beta,\hat{\beta}) + (\beta',\hat{\beta}') := (\sigma,\hat{\sigma})$ if $\overline{b,a} \neq \overline{b,c'}$ and $c \in \hat{\sigma}$,
$(\beta,\hat{\beta}) + (\beta',\hat{\beta}') := (\sigma,\check{\sigma})$ if $\overline{b,a} \neq \overline{b,c'}$ and $c \notin \hat{\sigma}$,
$(\beta,\hat{\beta}) + (\beta',\hat{\beta}') := (\sigma,[\ \overline{a,b}\ ;\ c\ ])$ if $\overline{b,a} = \overline{b,c'}$, where $[\ \overline{a,b}; c\ ]$ denotes the halfplane with edge the line $\overline{a,b}$ and containing the point c .

If $2R$ denotes two right angles, i.e., the *straight angle*, we shall write:

$(\beta,\hat{\beta}) + (\beta',\hat{\beta}') < 2R$ if $(\beta,\hat{\beta}) + (\beta',\hat{\beta}') = (\sigma,\hat{\sigma})$,
$(\beta,\hat{\beta}) + (\beta',\hat{\beta}') > 2R$ if $(\beta,\hat{\beta}) + (\beta',\hat{\beta}') = (\sigma,\check{\sigma})$,
$(\beta,\hat{\beta}) + (\beta',\hat{\beta}') = 2R$ if $\overline{b,a} = \overline{b,c'}$.

3. Lambert–Saccheri quadrangles

Consider, in the absolute plane $\mathbf{A} = (\mathcal{P}, \mathcal{L}, \alpha, \equiv)$, a Lambert–Saccheri quadrangle (shortly *L-S quadrangle*) (a, b, c, d) with $\delta := \angle(c, d, a)$.

Denoting by $a' := (a \perp \overline{c,d}) \cap \overline{c,d}$, we remember that the congruence $\equiv$ is called *singular* if $a' = d$, *hyperbolic* if $a' \in]c, d[$ (i.e., $(a' \mid c, d) = -1$) and *elliptic* if $d \in]a', c[$ (i.e., $(a' \mid c, d) = 1$).

For the following it will be useful to recall the well known result which is a consequence of [6], (16.9):

Proposition 3.1. *For any line $A \in \mathcal{L}$ and any two distinct points $b, c \in \mathcal{P} \setminus A$, if A and $\overline{b,c}$ have a common perpendicular $G \in \mathcal{L}$ then $(A \mid b, c) = 1$.*

The value of the angle δ is related to the congruence $\equiv$ defined in $\mathbf{A}$ by the following:

Theorem 1. *In any Lambert–Saccheri quadrangle (a, b, c, d), denoting by δ the angle $\angle(c, d, a)$, it holds:*

$$\delta = R \text{ if and only if the congruence} \equiv \text{is singular;}$$

$$\delta < R \text{ if and only if the congruence} \equiv \text{is hyperbolic;}$$

$$\delta > R \text{ if and only if the congruence} \equiv \text{is elliptic.}$$

Proof. Given the L-S quadrangle (a, b, c, d), let $A := \overline{a,d}$, $D := (d \perp \overline{c,d})$, $a' := \overline{c,d} \cap (a \perp \overline{c,d})$ and $a'' := D \cap (a \perp D)$, so $\epsilon := \angle(c, d, a'') = R$. Then by (3.1) $(\overline{d,c} \mid a, a'') = 1$ and if $a' \neq d$ also $(\overline{d,a''} \mid a, a') = 1$.

If $a' \in]c, d[$ then $(\overline{d,a''} \mid a', c) = (d \mid a', c) = 1$ and so $(\overline{d,a''} \mid a, c) = (\overline{d,a''} \mid a, a') \cdot (\overline{d,a''} \mid a', c) = 1 \cdot 1 = 1$, that means $a \in \hat{\epsilon}$, i.e., $\delta < \epsilon = R$.

If $d \in]a', c[$ then $(A \mid a', c) = (d \mid a', c) = -1$. Moreover (again by (3.1)) $(\overline{a',d} \mid a, a'') = (\overline{a'',d} \mid a, a') = 1$, hence (by (13.12.2) of [6]) $(A \mid a', a'') = -1$ and so we get $(A \mid a'', c) = (A \mid a', a'') \cdot (A \mid a', c) = -1 \cdot (-1) = 1$ and this means that $\delta > R$.

Since by [2], p. 162 and [5] the congruence is singular if and only if $\delta = R$, the theorem is proved. □

4. Triangles

By an *oriented triangle* $\triangle$ we understand (up to cyclic order) a tripel (a, b, c) of non collinear points – the *vertices* – together with the segments $]a, b[,]b, c[,]c, a[$ – the *sides* – and with the angles $\alpha := \angle(c, a, b)$, $\beta := \angle(a, b, c)$, $\gamma := \angle(b, c, a)$.

Let a', b', c' be respectively the midpoints of (b, c), (c, a), (a, b). Then $\alpha = \angle(b', a, c')$, $\beta = \angle(c', b, a')$, $\gamma = \angle(a', c, b')$, and, if $\widetilde{b'}, \widetilde{c'}$ are the point-reflections in the points b', c' respectively, we have

$$\gamma' := \widetilde{b'}(\gamma) = \angle\big(\widetilde{b'}(a'), a, b'\big) = \angle\big(\widetilde{b'}(b), a, c\big)$$

and

$$\beta' := \widetilde{c'}(\beta) = \angle\big(c', a, \widetilde{c'}(a')\big) = \angle\big(b, a, \widetilde{c'}(c)\big)\,.$$

Since $\widetilde{b'}$ and $\widetilde{c'}$ are proper motions the angles γ and $\gamma' = \widetilde{b'}(\gamma)$ are properly congruent and so are β and $\beta' = \widetilde{c'}(\beta)$. Since $\gamma' = \angle(\overrightarrow{a, \widetilde{b'}(b)}; \overrightarrow{a, c})$, $\alpha = \angle(\overrightarrow{a, c}; \overrightarrow{a, b})$ and $(\overline{a, c} \mid \widetilde{b'}(b), b) = (b' \mid \widetilde{b'}(b), b) = -1$, we can define:

$$\sigma_\gamma := \gamma' + \alpha = \angle\big(\overrightarrow{a, \widetilde{b'}(b)}; \overrightarrow{a, b}\big) = \angle\big(\widetilde{b'}(a'), a, b\big)\,.$$

Moreover, from $(b' \mid \widetilde{b'}(b), b) = -1$, it follows $(\widetilde{b'}(b) \mid b, b') = (b \mid b', \widetilde{b'}(b)) = 1$ and from $\widetilde{b'}(a) = c$ it follows $(b' \mid a, c) = -1$ and so $(a \mid b', c) = (c \mid b', a) = 1$. Therefore

$$\big(\overline{a, \widetilde{b'}(b)} \mid b, c\big) = \big(\overline{a, \widetilde{b'}(b)} \mid b', c\big)\cdot\big(\overline{a, \widetilde{b'}(b)} \mid b', b\big) = (a \mid c, b')\cdot\big(\widetilde{b'}(b) \mid b', b\big) = 1\cdot 1 = 1\,,$$

and

$$\big(\overline{a, b} \mid c, \widetilde{b'}(b)\big) = (\overline{a, b} \mid c, b') \cdot \big(\overline{a, b} \mid b', \widetilde{b'}(b)\big) = (a \mid c, b') \cdot \big(b \mid b', \widetilde{b'}(b)\big) = 1 \cdot 1 = 1\,.$$

Consequently:

$$(\gamma', \widehat{\gamma'}) + (\alpha, \hat{\alpha}) = (\sigma_\gamma, \widehat{\sigma_\gamma})\,.$$

In the same way, for

$$\sigma_\beta := \alpha + \beta' = \angle\big(\overrightarrow{a, c}; \overrightarrow{a, \widetilde{c'}(c)}\big) = \angle\big(b, a, \widetilde{c'}(c)\big)$$

it holds

$$(\alpha, \hat{\alpha}) + (\beta', \widehat{\beta'}) = (\sigma_\beta, \widehat{\sigma_\beta})\,.$$

Moreover:

$$\begin{aligned}\big(\overline{a, b} \mid \widetilde{b'}(b), \widetilde{c'}(c)\big) &= \big(\overline{a, b} \mid \widetilde{b'}(b), b'\big) \cdot (\overline{a, b} \mid b', c) \cdot \big(\overline{a, b} \mid c, \widetilde{c'}(c)\big)\\ &= \big(b \mid \widetilde{b'}(b), b'\big) \cdot (a \mid b', c) \cdot \big(c' \mid c, \widetilde{c'}(c)\big) = 1 \cdot 1 \cdot (-1) = -1\end{aligned}$$

and in the same way

$$\big(\overline{a, c} \mid \widetilde{b'}(b), \widetilde{c'}(c)\big) = -1\,.$$

Therefore we can form $\sigma_\gamma + \beta'$ and $\gamma' + \sigma_\beta$ and obtain

$$\sigma_{\triangle,a} := \sigma_\gamma + \beta' = \gamma' + \sigma_\beta = \angle\big(\overrightarrow{a, \widetilde{b'}(b)}; \overrightarrow{a, \widetilde{c'}(c)}\big) = \angle\big(\widetilde{b'}(b), a, \widetilde{c'}(c)\big)\,.$$

We call $\sigma_{\triangle,a}$ the *sum of the angles of the triangle* $\triangle$ *in the vertex* a. In the same way we have $\widetilde{b'}(\sigma_{\triangle,a}) = \sigma_{\triangle,c}$ and $\widetilde{c'}(\sigma_{\triangle,a}) = \sigma_{\triangle,b}$ and so the sums in the three vertices are properly congruent.

Now we are ready to show that the sum of the angles of $\triangle$ is equal, smaller or greater than $2R$ according to the properties of the congruence of the general absolute plane **A**. Namely we prove the following theorem which, since a Euclidean plane is singular, can be considered as an extension to a general absolute plane of the classical results recalled in (**1.1**) and stated by the *"first Legendre theorem"*:

Theorem 2. *For a general absolute plane* **A** *let* $\triangle = (a, b, c)$ *be any oriented triangle,* a', b', c' *be respectively the midpoints of* $(b, c), (c, a), (a, b)$ *and* $\sigma_{\triangle,a}$ *be the sum of the interior angles* $(\gamma', \widehat{\gamma'})$, $(\alpha, \widehat{\alpha})$ *and* $(\beta', \widehat{\beta'})$. *Then exactly one of the following statements is fulfilled:*

1. **A** *is singular if and only if* a *is the midpoint of* $\widetilde{b'}(b)$ *and* $\widetilde{c'}(c)$, *i.e.,*

$$\sigma_{\triangle,a} = 2R, \text{ i.e., } \sigma_{\triangle,a} \text{ is a straight angle};$$

2. **A** *has a hyperbolic congruence if and only if the sum of the interior angles is the interior angle* $(\sigma_{\triangle,a}, \widehat{\sigma}_{\triangle,a})$, *i.e.,*

$$\left(\gamma', \hat{\gamma'}\right) + (\alpha, \hat{\alpha}) + \left(\beta', \hat{\beta'}\right) = (\sigma_{\triangle,a}, \widehat{\sigma}_{\triangle,a}) < 2R;$$

3. **A** *has an elliptic congruence if and only if the sum of the interior angles is the exterior angle* $(\sigma_{\triangle,a}, \check{\sigma}_{\triangle,a})$, *i.e.,*

$$\left(\gamma', \hat{\gamma'}\right) + (\alpha, \hat{\alpha}) + \left(\beta', \hat{\beta'}\right) = (\sigma_{\triangle,a}, \check{\sigma}_{\triangle,a}) > 2R.$$

Proof. In the oriented triangle $\triangle = (a, b, c)$ let $A := \overline{b, c}$, $M := \overline{b', c'}$, $\bar{b} := (b \perp M) \cap M$, $\bar{c} := (c \perp M) \cap M$, $\bar{a} := (a \perp M) \cap M$, $m := (a' \perp M) \cap M$, A' the midline of b, c and $A" := (a \perp M)$. By [6], (18.10) we know :

(i) $M \perp A' \perp A$, hence $m \in A'$, $(\bar{b}, m, a', b)$ and $(\bar{c}, m, a', c)$ are L-S quadrangles and $\widetilde{A'}(b) = c$ implying $\tilde{m}(\bar{b}) = \widetilde{A'}(\bar{b}) = \bar{c}$, i.e., the line reflection $\widetilde{A'}$ interchanges the two L-S quadrangles .

From (i) and $\widetilde{b'}(a) = c$, $\widetilde{c'}(a) = b$ follows:

(ii) $\widetilde{b'}(\bar{a}) = \bar{c}$, $\widetilde{c'}(\bar{a}) = \bar{b}$ hence $\widetilde{b'} \circ \widetilde{m} \circ \widetilde{c'}(\bar{a}) = \bar{a}$.

Since $m, b', c', \bar{a} \in M$, by [6], (17.13.2), the product of the point reflections $\widetilde{b'} \circ \widetilde{m} \circ \widetilde{c'}$ is again a point reflection fixing the point $\bar{a}$ (by (ii)) and therefore:

(iii) $\widetilde{b'} \circ \widetilde{m} \circ \widetilde{c'} = \widetilde{\bar{a}} = \widetilde{c'} \circ \widetilde{m} \circ \widetilde{b'}$ hence $\widetilde{\bar{a}} \circ \widetilde{b'} = \widetilde{c'} \circ \widetilde{m}$ and so $\widetilde{\bar{a}}(\widetilde{b'}(m)) = \widetilde{c'}(\widetilde{m}(m)) = \widetilde{c'}(m)$. Moreover by $A" \perp M$ and $\bar{a} = A" \cap M$, $\widetilde{A"}(\widetilde{b'}(m)) = \widetilde{\bar{a}}(\widetilde{b'}(m)) = \widetilde{c'}(m)$ and so $(A" \mid \widetilde{b'}(m), \widetilde{c'}(m)) = -1$.

From (i) follows

(iv) $A = \overline{a', c} \perp A' = \overline{a', m} \perp M$ implies

$$X := \widetilde{b'}(A) = \overline{\widetilde{b'}(a'), \widetilde{b'}(c)} = \overline{\widetilde{b'}(a'), a} \;\perp \widetilde{b'}(A') = \overline{\widetilde{b'}(a'), \widetilde{b'}(m)} \;\perp M$$

and $A = \overline{a', b} \perp A' = \overline{a', m} \perp M$ implies

$$Y := \widetilde{c'}(A) = \overline{\widetilde{c'}(a'), \widetilde{c'}(b)} = \overline{\widetilde{c'}(a'), a} \;\perp \widetilde{c'}(A') = \overline{\widetilde{c'}(a'), \widetilde{c'}(m)} \;\perp M.$$

Now we apply the point reflections $\widetilde{b'}$ and $\widetilde{c'}$ firstly on the L-S quadrangles $(\bar{c}, m, a', c)$ and $(\bar{b}, m, a', b)$ resp. (cf., (i)) and obtain by (ii) and (iii),

(v) $(\bar{a}, \widetilde{b'}(m), \widetilde{b'}(a'), a)$ and $(\bar{a}, \widetilde{c'}(m), \widetilde{c'}(a'), a)$ are congruent L-S quadrangles with $A" = \overline{a, \bar{a}}$ and $\widetilde{A"}(\bar{a}, \widetilde{b'}(m), \widetilde{b'}(a'), a) = (\bar{a}, \widetilde{c'}(m), \widetilde{c'}(a'), a)$ hence $(A" \mid \widetilde{b'}(a'), \widetilde{c'}(a')) = -1$. Applying now $\widetilde{b'}$ and $\widetilde{c'}$ on the points b, a', c (a' is the midpoint of b and c) we obtain:

(vi) $\widetilde{b'}(a')$ is the midpoint of a and $\widetilde{b'}(b)$, $\widetilde{c'}(a')$ is the midpoint of a and $\widetilde{c'}(c)$ implying $\overrightarrow{a,\widetilde{b'}(a')} = \overrightarrow{a,\widetilde{b'}(b)}$ and $\overrightarrow{a,\widetilde{c'}(a')} = \overrightarrow{a,\widetilde{c'}(c)}$ and so $\sigma_{\triangle,a} := \sigma_\gamma + \beta' = \gamma' + \sigma_\beta = \angle(\widetilde{b'}(b), a, \widetilde{c'}(c)) = \angle(\widetilde{b'}(a'), a, \widetilde{c'}(a'))$.

By (v) and (vi) we have:

(vii) $\widetilde{A"}(\widetilde{b'}(b)) = \widetilde{c'}(c))$.

(viii) Denoting by $L := \overline{\widetilde{b'}(a'), \widetilde{c'}(a')}$ we have $L \perp A"$ hence there exists a point $l := L \cap A"$ with $(l \mid \widetilde{b'}(a'), \widetilde{c'}(a')) = (A" \mid \widetilde{b'}(a'), \widetilde{c'}(a')) = -1$ (cf., (v)).

Thus we obtain the following:

1. **A** is singular if and only if $l = a$ (i.e., the two congruent L-S quadrangles in (v) are rectangles); this means that a is the midpoint of $\widetilde{b'}(a')$ and $\widetilde{c'}(a')$, or equivalently a is the midpoint of $\widetilde{b'}(b)$ and $\widetilde{c'}(c)$ (cf., (vii)), i.e., $\sigma_{\triangle,a}$ is a straight angle.
2. **A** is ordinary if and only if $l \neq a$ and, by [5], the congruence $\equiv$ is hyperbolic if and ony if $l \in]a, \bar{a}[$, i.e., $(l \mid a, \bar{a}) = -1$ and so $(a \mid \bar{a}, l) = (\bar{a} \mid l, a) = 1$. Since $c' \in M$, by (i) and (iv) we have $(X \mid \bar{a}, c') = (Y \mid \bar{a}, c') = 1$, $(X \mid l, \widetilde{c'}(a')) = (\widetilde{b'}(a') \mid l, \widetilde{c'}(a')) = 1 = (Y \mid l, \widetilde{b'}(a'))$ and so

$$\big(X \mid c', \widetilde{c'}(a')\big) = (X \mid c', \bar{a}) \cdot (X \mid \bar{a}, l) \cdot \big(X \mid l, \widetilde{c'}(a')\big) = 1 \cdot (a \mid \bar{a}, l) \cdot 1 = (a \mid \bar{a}, l)\,,$$

but the same holds for $(Y \mid c', \widetilde{b'}(a'))$, so we have obtained:

$$\big(X \mid c', \widetilde{c'}(a')\big) = (a \mid \bar{a}, l) = \big(Y \mid c', \widetilde{b'}(a')\big)\,.$$

Since $\gamma' := \angle\big(\widetilde{b'}(a'), a, b'\big)$ and $\beta' := \angle\big(c', a, \widetilde{c'}(a')\big)$ we have

$$\gamma' + \alpha + \beta' = \sigma_{\triangle,a} = \angle\big(\widetilde{b'}(a'), a, \widetilde{c'}(a')\big)$$

thus, by the definition of Section 2, it follows

$$\big((\gamma', \hat{\gamma}') + (\alpha, \hat{\alpha})\big) + \big(\beta', \hat{\beta}'\big) = (\sigma_{\triangle,a}, \widehat{\sigma}_{\triangle,a}) < 2R$$

if and only if $c' \in \widehat{\sigma}_{\triangle,a}$ (hence also $b' \in \widehat{\sigma}_{\triangle,a}$), i.e., $(X \mid c', \widetilde{c'}(a')) = (Y \mid c', \widetilde{b'}(a')) = 1$ (or respectively $(X \mid b', \widetilde{c'}(a')) = (Y \mid b', \widetilde{b'}(a')) = 1$). This is the case if and only if the congruence $\equiv$ is hyperbolic.

Finally, the characterization 3. of the elliptic congruence follows directly from 1. and 2. □

5. Hyperbolic planes

Now we assume that **A** is a general hyperbolic plane, i.e., the following *hyperbolic parallel axiom* **H** holds true (cf., [6], p. 149):

H – For any line G and any point a not on G, there exists at least one line H through a which is *hyperbolic parallel* (shortly *h-parallel*) to G, namely such that the following conditions are fulfilled:

H1 – $H \cap G = \emptyset$.

H2 – The line A through the point a and perpendicular to G, is not perpendicular to H.

H3 – Denoting by $\widetilde{A}$ the reflection in the line A, for any x not on $(H \cup \widetilde{A}(H))$, if $(H \mid x, \widetilde{A}(x)) = +1$ then $\overline{a,x} \cap G \neq \emptyset$.

In a hyperbolic plane $\mathbf{A}$, an *end* $\mathfrak{e}$ is defined as a pencil of lines such that any two distinct lines of the pencil are mutually h-parallel (cf., [6], § 27). By [6], (27.2) any line is contained in exactly two ends.

Theorem 3. *The congruence of a general hyperbolic plane is hyperbolic.*

Proof. Let $A \in \mathcal{L}$, $a \in A$ and let $\mathfrak{a}$ be one of the ends containing the line A. Let $B := (a \perp A)$, $b \in B \setminus \{a\}$, $C := (b \perp B)$ and let $D := \mathfrak{a} \cap \tilde{C}(\mathfrak{a})$ (cf., [6], (27.3.1) and (27.8)). Then $D \perp C$ and $c := D \cap C$ exists. Now let $A' := (c \perp A)$, $a' := A' \cap A$, $A'' := (a' \perp C)$ and $a'' := A'' \cap C$. Then (a', a, b, c) is a Lambert–Saccheri quadrangle and we have: $(c \mid a'', b) = (D \mid a'', b) = (D \mid a'', a') \cdot (D \mid a', a) \cdot (D \mid a, b)$. Since D and A'' and also D and B have the common perpendicular C , by (3.1) we have $(D \mid a'', a') = (D \mid a, b) = 1$ and by $A \cap D = \emptyset$ (since A and D are hyperbolic parallel) also $(D \mid a', a) = 1$ thus $(c \mid a'', b) = 1$. In the same way, $(b \mid a'', c) = (B \mid a'', c) = (B \mid a'', a') \cdot (B \mid a', c) = 1 \cdot 1 = 1$, since B and A'' have the common perpendicular C and B and A' have the common perpendicular A. Now $(c \mid a'', b) = (b \mid a'', c) = 1$ implies $(a'' \mid c, b) = -1$. Therefore the congruence is hyperbolic. □

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