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Groupoids Associated with the Ternary Ring of a Projective Plane

Dedicated to Helmut Karzel on the occasion of his 70th birthday

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0. INTRODUCTION

Let π be a projective plane [1, 6] with the incidence relation I and coordinatized by a ternary ring (R,F) chosen in π relative to a coordinatizing quadrangle (X, Y, Q, E). The points of π are the elements (x,y) ($x, y \in R$) esp. Q = (0,0) E = (1,1), the (m) ($m \in R$), esp. X = (0), and Y ; the lines of π are the [m,k] ($m, k \in R$), [c] ($c \in R$), L; the incidence relation I is determined by: $y = F(x,m,k) \Leftrightarrow (x,y)\ I\ [m,k]$, $\forall\, x,m,k \in R$; (x,y) I [x]; (x,x) I [1,0]; (m) I [m,k]; (m) I L; (0,k) I [m,k]; (1,m) I [m,0]; Y I L. Using the notation of [3] the intersection of two different lines a, b is denoted by $a \cap b$, the join of two different points A, B by [AB], while by (A B C) we denote the collinearity of the points A, B, C. Using the two elements 0, 1 (0, 1 are two distinct elements in the coordinatizing set R) three binary operations, namely, +, $\cdot$, * were defined out of the ternary ring (R,F) (see [1,p. 50], [2] and [3]) as follows: $a+b = F(a, 1, b)$, $a \cdot b = F(a, b, 0)$, $a*b = F(1, a, b)$, $\forall\, a,b \in R$. A fourth binary operation, denoted by "o" has been defined as follows [3]:

$$a \circ b = F(a,b,1),\ \forall\, a,b \in R.$$

In this paper we introduce on (R,F) three binary operations by fixing any element p in R as follows:

$$a o_p b = F(a,b,p),\ \forall\ a,b \in R,$$

$$a +_p b = F(a,p,b),\ \forall\ a,b \in R;\ p \neq 0,$$

$$a *_p b = F(p,a,b),\ \forall\ a,b \in R;\ p \neq 0.$$

We shall develop some algebraic and geometric properties of the groupoids (binary systems) (R,o_p), $(R,+_p)$ and $(R,*_p)$ in the presence of certain configurational propositions [7, p. 22-23].

1. THE GROUPOID (R,o_p)

Let p be any fixed element in R. Then, for any $a,b \in R$, we define $a o_p b = F(a,b,p)$. Thus (R,o_p) is a binary system (a groupoid). We write $a o_0 b = a \cdot b$ and $a o_1 b = a o b$.

LEMMA 1. For any $a,b \in R$, $a o_p b = p \Leftrightarrow a = 0$ or $b = 0$.

Proof. $a o_p b = p \Leftrightarrow F(a,b,p) = p \Leftrightarrow (a, p)I[b,p] \Leftrightarrow [a], [0, p], [b, p]$ are concurrent $\Leftrightarrow (0, b)I[a]$ or $[0, p] = [b, p] \Leftrightarrow a = 0$ or $b = 0$. □

LEMMA 2. For any $a,b \in R$, the two equations $a o_p x = b$ and $y o_p a = b$ have unique solutions in R if, and only if, $a \neq 0$.

Proof. Let $a,b \in R$; $a \neq 0$, then $a o_p x = b \Leftrightarrow F(a,x,p) = b \Leftrightarrow (a,b)I[x,p] \Leftrightarrow (x) = L \cap [(a,b)(0,p)]$, which shows that there exists exactly one solution x in R. Similarly, $y o_p a = b \Leftrightarrow F(y,a,p) = b \Leftrightarrow (y,b)I[a,p] \Leftrightarrow (y,b) = [0,b] \cap [a,p]$, which shows that there exists exactly one solution y in R. By Lemma 1, there are no solutions in the case $a = 0$ for $b \neq p$ and every element of R is a solution if $b = p$. □

REMARK. From Lemma 2 follow the following two cancellation laws in the groupoid (R,o_p):
(i) $a o_p b = a o_p c \Rightarrow b = c,\ \forall\ a,b,c \in R;\ a \neq 0$,
(ii) $b o_p a = c o_p a \Rightarrow b = c,\ \forall\ a,b,c \in R;\ a \neq 0$.

It follows, from Lemma 2, that (R,o_p) is not a quasigroup (in particular $(R, \cdot)$ and (R, o) are not quasigroups). We also note that if $p = 0$, i.e., $a o_p b = a \cdot b$, then $(R\backslash\{0\},\cdot)$ is a loop [1, p.50]. But if $p \neq 0$ (esp. $p = 1$), then the equation $po_px = 0$ has a unique solution x ,which is $\neq 0$, since $po_p0 = p$. Also, the equation $yo_pp = 0$, has a unique solution y, which is $\neq 0$, since $0o_pp = p$. Hence, the pair $(R\backslash\{0\}, o_p)$ is not a groupoid, i.e., the exclusion of 0 doesn't help in the case $p \neq 0$ (esp. $p = 1$). Moreover, in the system (R, o_p), $p \neq 0$, there is no right or left identity element because the two equations $xo_p0 = p$ and $0o_py = p$ hold true for any $x, y \in R$. The following lemma shows that the exclusion of 0 again doesn't help (for the case $p = 1$, compare [3]).

LEMMA 3. Let p be a fixed element in $R\backslash\{0\}$(esp. $p = 1$). Then, unless $R = \{0,1\}$, there is no $x, y \in R$ such that:

(i) $xo_pa = a, \forall a \in R\backslash\{0\}$,

(ii) $ao_py = a, \forall a \in R\backslash\{0\}$.

Proof. For $a = p$, in equation (i) as well as (ii), we obtain $x = 0$, $y = 0$, respectively. Now for any $a \neq p$, we have $0o_pa = a \Rightarrow a = p$, also, $ao_p0 = a \Rightarrow a = p$. In particular for $a = 1$, we get $p = 1$ and hence $R = \{0,1\}$. Conversely, if $R = \{0,1\}$,i.e., $p = 1$, the two equations $xo1 = 1$ and $1oy = 1$, have, by Lemma 1, the solution $x = y = 0$. □

LEMMA 4. For $p \neq 0$, the groupoid (R, o_p) is non-associative in any ternary ring (R,F).

Proof. $0o_p(1o_p1) = p$, but $(0o_p1)o_p1 = po_p1 = F(p, 1, p) \neq p$ since (p, p) ($\neq Q$) is on $[1, 0]$ and therefore not on $[1, p]$. □

However, a weak form of associativity, namely $(ao_pa)o_pa = ao_p(ao_pa)$, $\forall a \in R$, may hold in those projective planes in which the following condition, denoted by C_1, is satisfied.

The condition C_1.

If 1, 2, 3, 4 are four points, no three of which are collinear , 0 a point not on the lines [1 2], [1 3], [1 4], [2 3], [2 4] and $3' = [1\ 3] \cap [2\ 4]$, $4' = [1\ 4] \cap [2\ 3]$, $5 = [0\ 3] \cap [1\ 2]$, $0' = [0\ 2] \cap [4\ 5]$, $6 = [0\ 4'] \cap [1\ 2]$, $7 = [0\ 3'] \cap [1\ 4]$, $8 = [0'6] \cap [2\ 4]$, $9 = [0'4] \cap [2\ 7]$, then (1 8 9).

THEOREM 1. C_1 holds, in a projective plane if, and only if, $(ao_pa)o_pa = ao_p(ao_pa)$, $\forall a \in R$ in every coordinatizing (R, F).

Proof: In the incomplete C_1-configuration set (X, Y, Q, E) so that (see Fig. 1): 1= X, 2= Y, 0 = Q and E = [2 3]$\cap$[[0 3´], (1,a) = 3, (a,b) = 4 with b = ao_pa. Thus, 3´= (a,a) , 4´= (1,b), 5 = (a), 0´= (0,p), 6 = (b), 7= (b,b), 8 = (a,ao_pb), 9 = (b,bo_pa) and hence (1 8 9) $\Leftrightarrow$ $ao_pb = bo_pa$. □

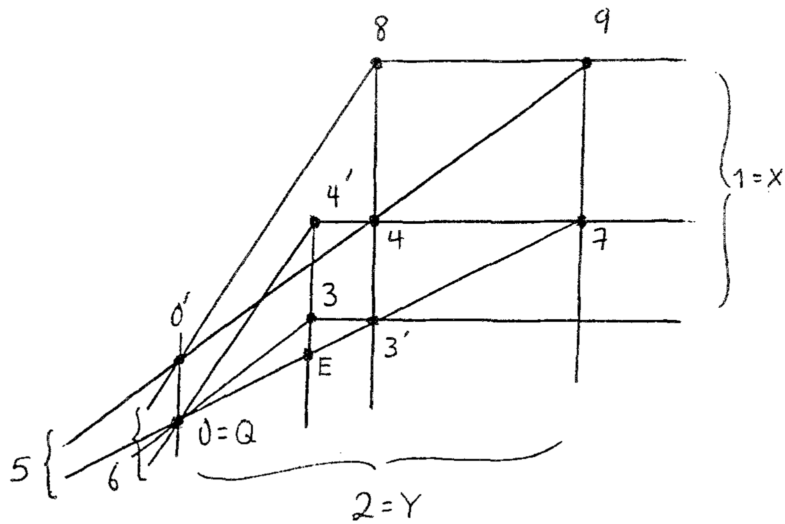

Fig. 1

It is to be noted that with the condition (0 3 4), C_1 is nothing else than the hexagon condition [6, p. 54] for the (0, 1, 2) - net. This condition for all non - collinear point triples 0, 1, 2 is equivalent with $a^2 . a = a . a^2$. We remark also that if (R,F) is linear ternary ring (i.e., F(x,m,k) = x.m+k), then $1o_pa$ = F(1,a,p) = a*p = a+p = F(a,1,p) = ao_p1. We write $\{(1,2,3), (1',2',3')\}^0(4\ 5\ 6)$ for two triangles (1,2,3), (1´,2´,3´) perspective from a point 0 and from a line [4 5]=[4 6]. An incomplete configuration [3] is a configuration with one missing incidence. We denote by D_1 the little Desargues proposition in which the center of perspectivity of D_1 is incident with the axis of perspectivity [4, p. 330]. The following theorem shows that the little Desargues proposition is the geometric representation of the algebraic identity $1o_pa = ao_p1$ (equivalently, a*p = a+p).

THEOREM 2. In any projective plane π the proposition D_1 holds if, and only if, $1o_pa = ao_p1$, $\forall$ a $\in$ R; p $\in$ R\ {0} for every coordinatizing (R, F).

Proof. Let (1,2,3), (1′,2′,3′) be two triangles generating an incomplete D_1-configuration in which {(1,2,3), (1′,2′,3′)}^{0} and (0 4 5) (see Fig. 2). Choose the coordinatizing quadrangle (X, Y, Q, E) so that X= [2 3]∩[4 5], 0= Y, 1 = Q, E = [1 3]∩[0 2], 2 = (1,a) and 1′ = (0,p) ; p≠ 0, thus 3 = (a,a), 2′= (1, F(1,a,p)), 3′= (a, F(a,1,p)) and therefore (2′3′X) ⇔ (6 I [4 5]) ⇔ F(1,a,p) = F(a,1,p). □

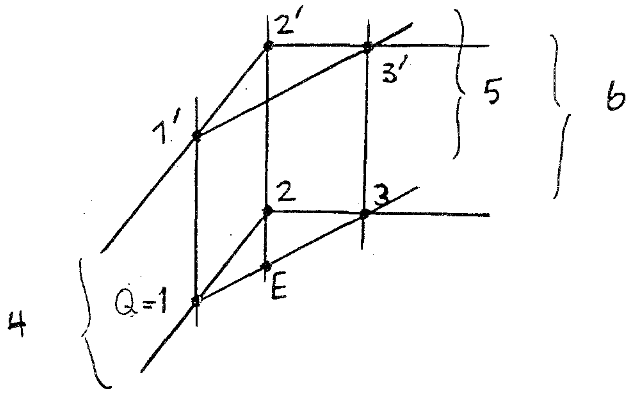

Fig. 2

It is to be noted that if p = 0, then 1·a = a·1 = a, ∀ a ∈ R, and if p = 1, then a special case of the little Desarques is shown to be the configurational representation of the algebraic identity 1oa = ao1, ∀ a ∈ R, (see theorem 1 in [3]).

Let (1, 2, 3) and (1′,2′,3′) be any two triangles in a projective plane π with 4 = [1 2]∩ [1′2′], 5 = [1 3]∩[1′3′], 6 = [2 3]∩[2′3′]. Special forms of Desargues proposition denoted by D_1(the little Desargues proposition), D_2 and D_3 arise when one requires one, two and three vertices of one triangle to lie on the sides of the other triangle [4, p.330]. We may reformulate D_2 and D_3 as follows:

The proposition D_2.
If {(1,2,3), (1′,2′,3′)}^{0}; with 1I[2′ 3′] and 1′I[2 3], then (4 5 6).

The proposition D_3.
If {(1,2,3), (1′,2′,3′)}^{0}; with 3I[1′2′] and 5I[2′2] and 6I[1′1] then (4 5 6) (see Fig. 6).

Now, a special form of the proposition D_2 may be formulated as follows:

The proposition $(D_2)^*$.
If the triangles (1, 2, 3) and (1´,2´,3´) are perspective from 0, 1´I[2 3], 1I[2´3´], 4 = [1 2]$\cap$ [1´2´] , 5 = [1 3]$\cap$[1´3´], 6 = [2 3]$\cap$[2´3´] and the lines [1 2], [1´3´], [0 6] are concurrent , then (4 5 6) (see Fig. 3).

THEOREM 5. $(D_2)^*$ is valid in a projective plane if , and only if, in every coordinatizing (R, F) of the plane the element a determined by ao1 = 0 fulfils also the equation 1oa = 0.

Proof. Let (1,2,3), (1´,2´,3´) be two triangles generating an incomplete $(D_2)^*$-configuration as depicted in Fig. 3. Assume that ao1 = 0, then we may set 6 = (0), 1 = Y, 1´ = Q and 7 = (1,1). It follows that 3´= (1), 0 = (0,1), 2 = (1,0) and (x,0)I[1,1] $\Rightarrow$ xo1 = 0 $\Rightarrow$ x = a.

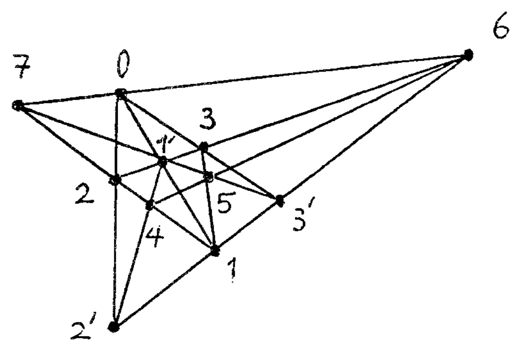

Fig. 3

Therefore, 5 = (a,a), 4 = (1,a) and 2´= (a). Now, the configuration is complete $\Leftrightarrow$ (0 2´2) holds true $\Leftrightarrow$ (1,0)I[a,1] $\Leftrightarrow$ 1oa = 0. □

The question which presents itself is: under what condition is the groupoid (R,o_p) commutative ? the question is answered for the two special cases: p = 0 and p = 1. In fact, it has been shown that the proposition of Pappus charcterizes the commutativity of multiplication "$\cdot$" [7, p.39] and the commutativity of the operation "o" [3, p.6]. Now a general answer is given in the following theorem.

THEOREM 6. The proposition of Pappus holds in a projective plane if, and only if, $a o_p b = b o_p a$, $\forall$ a,b $\in$ R; p $\in$ R in all coordinatizing (R, F) of the plane.

Proof. The result follows by observing that the proof of Theorem 16 in [5] remains valid by putting c = p, i.e., when we set 1′= (0,p) and E = (1,1) I[1 2]. □

The relationships of the two configurational propositions D_1 and D_2 with the groupoid (R, o_p) are explained in the following sequence of theorems and their corollaries.

THEOREM 7. D_1 holds in a projective plane if, and only if, in every coordinatizing ternary ring (R,F), $a\cdot b = c\cdot d \Rightarrow a o_p b = c o_p d$, $\forall$ a,b,c,d $\in$ R; p $\in$ R\{0}.

Proof: Let (1, 2, 3) and (1′, 2′, 3′) be two triangles generating an incomplete D_1-configuration with center of perspectivity 0 and (0 4 5). Putting (see Fig. 4) 0= Y, 2′= Q, 2 = (0,p); $p \neq 0$, [3′3]= [a], 4= (d), 5= (b), X= [4 5]∩[1′3′], thus 3′= [a]∩[b,0] = (a,a$\cdot$b), 1′= [d,0]∩[0,a$\cdot$b] = (c,a$\cdot$b), 1= (c,F(c,d,p)), 3= (a,F(a,b,p)) and hence (4 5 6) ⇔ (1 3 X)⇔ F(c,d,p) = F(a,b,p). □

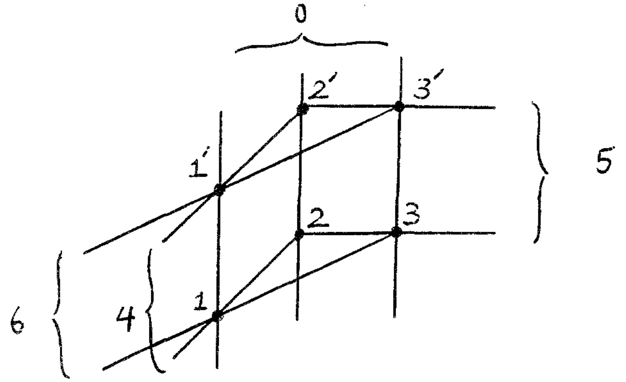

Fig.4

COROLLARY 1. D_1 holds in a projective plane if, and only if, in every coordinatizing ternary ring (R,F), $a o_p b = (a\cdot b) o_p 1$, $\forall$ a,b $\in$ R; p $\in$ R\{0}.

Proof: This result follows from the fact that the two algebraic identities given in Theorem 7 and the preceding corollary are equivalent. To establish this we suppose first that: for any $a,b,c,d \in R$, $a \cdot b = c \cdot d \Rightarrow a o_p b = c o_p d$. Then putting $d = 1$, we obtain $a \cdot b = c \Rightarrow a o_p b = c o_p 1 = (a \cdot b) o_p 1$. For the converse, assume that: $a o_p b = (a \cdot b) o_p 1$, $\forall\, a,b \in R$. Then, $a \cdot b = c \cdot d \Rightarrow (a \cdot b) o_p 1 = (c \cdot d) o_p 1 \Rightarrow a o_p b = c o_p d$. □

THEOREM 8. D_2 holds in a projective plane if, and only if, in every coordinatizing (R, F) $a \cdot b = c \cdot d = p \Rightarrow a o_p b = c o_p d$, $\forall\, a,b,c,d \in R$; $p \in R \backslash \{0\}$.

Proof: Follow the proof of Theorem 13 in [5] by setting $e = p$.□

When $E = (1,1)$, in the proof of Theorem 13 in [5], is restricted to be incident with $[1'2']$, we obtain the the following result.

COROLLARY 2. D_2 holds in a plane if, and only if, in every (R,F) $a \cdot b = c \cdot d = 1 \Rightarrow a o b = c o d$, $\forall\, a,b,c,d \in R$.

THEOREM 9. D_2 holds in a plane if, and only if, in every coordinatizing (R, F) $a \cdot b = p \Rightarrow a o_p b = p o_p 1$, $\forall\, a,b \in R$; $p \in R \backslash \{0\}$.

Proof. Since the two algebraic conditions of Theorem 8 and Theorem 9 are equivalent, we conclude this result. □

Similarly, the following result is a consequence of Corollary 2.

COROLLARY 3. D_2 holds in a plane if, and only if, $a \cdot b = 1 \Rightarrow a o b = 1 o 1$, $\forall\, a,b,c,d \in R$, for every coordinatizing (R, F).

Finally, the relationship of the proposition $\overline{F}$, which states that the diagonal points of a complete quadrangle form a collinear triple [4, p. 329], with the groupoid (R, o_p); $p \neq 0$, is explained in the following theorem.

THEOREM 10. The proposition $\overline{F}$ holds, in any projective plane π, if, and only if, in every coordinatizing (R, F) of π, $a \cdot b = p \Rightarrow a o_p b = 0$, $\forall\, a,b \in R$; $p \in R \backslash \{0\}$.

Proof. Observing the validity of the proof of Theorem 8 in [5], when c is taken to be any element p in $R\backslash\{0\}$, this result can be established. □

For the special case p = 1, i.e., E = (1,1)I[2 4], we state the following result obtained also in [3, p. 8].

COROLLARY 4. The proposition $\bar{F}$ holds in π if, and only if, in every coordinatizing (R, F), $a\cdot b = 1 \Rightarrow a o b = 0$, $\forall$ a,b $\in$ R.

Finally, we generalize the results obtained in the two theorems 6, 7, given in [3], to be satisfied in any groupoid (R, o_p); such that p is taken to be any fixed element in $R\backslash\{0\}$. In fact, we prove analogously the following theorem.

THEOREM 11. In any groupoid (R, o_p); $p \neq 0$, the following properties are equivalent:

(1) $a o_p b = (a\cdot b) o_p 1$, $\forall$ a,b $\in$ R,

(2) $a\cdot b = c\cdot d \Rightarrow a o_p b = c o_p d$, $\forall$ a,b,c,d $\in$ R,

(3) $a o_p b = c o_p d \Rightarrow a\cdot b = c\cdot d$, $\forall$ a,b,c,d $\in$ R,

(4) $\exists f_p, g_p: R \to R$; $a o_p b = f_p(a\cdot b) + g_p(a+b)$, $\forall$ a,b $\in$ R,

(5) $\exists \varphi_p, \psi_p: R \to R$; $a\cdot b = \varphi_p(a o_p b) + \psi_p(a+b)$, $\forall$ a,b $\in$ R.

Proof. For (1) $\Leftrightarrow$ (2), see the proof of Corollary 1. Now to show (1) $\Leftrightarrow$ (3), we assume first (1) holds true. Then, $a o_p b = c o_p d \Rightarrow (a\cdot b) o_p 1 = (c\cdot d) o_p 1$, and hence by Lemma 2, we get $a\cdot b = c\cdot d$, i.e., (3) holds true. Suppose (3) holds true. Then, by Lemma 2, the equation $x o_p 1 = a o_p b$ has unique solution in R and by (3), it follows that $x = x\cdot 1 = a\cdot b$, i.e., $a o_p b = (a\cdot b) o_p 1$.Thus (1) holds true. Now we show (1) implies (4). In fact, we let: $f_p(x) := x$ and $g_p(x) := p$, $\forall$ x $\in$ R. By (1), for any a,b $\in$ R, we have $a o_p b = (a\cdot b) o_p 1 = a\cdot b + p$.Thus, $a o_p b = f_p(a\cdot b) + g_p(a+b)$. Hence (4) holds true. For the converse, we suppose (4). Putting a = 0 in (4), then $p = f_p(0) + g_p(b)$, $\forall$ b $\in$ R. Since (R,+) is a loop, it follows that there exists k $\in$ R with $g_p(x) := k$, $\forall$ x $\in$ R. Since $a\cdot b = (a\cdot b)\cdot 1$, we have

$$\begin{aligned} a o_p b &= f_p(a\cdot b) + g_p(a+b) \\ &= f_p(a\cdot b) + k \\ &= f_p((a\cdot b)\cdot 1) + k \\ &= (a\cdot b) o_p 1. \end{aligned}$$

Thus (4) $\Rightarrow$ (1). We conclude this proof by showing (1) $\Leftrightarrow$ (5). Suppose that (1) holds true, i.e., $a o_p b = (a\cdot b) o_p 1$, for any a,b in R. Then we may put $\varphi_p(x) := y$ such that y is the unique solution of the equation $y+p = x$. Also, put $\psi_p(x) := 0$. Thus, from $a\cdot b = \varphi_p(a\cdot b+p)$ and $a\cdot b+p = ao_pb$ (by (1)), we obtain that

$$\begin{aligned} a\cdot b &= \varphi_p(ao_pb) + 0 \\ &= \varphi_p(ao_pb) + \psi_p(a+b). \end{aligned}$$

Hence (5) is valid.

Remains to show that: (5) $\Rightarrow$ (1). Now putting $a = 0$ in (5), we obtain

$$0 = \varphi_p(p) + \psi_p(b),\ \forall\ b \in R.$$

Consequently, since $(R,+)$ is a loop, there exists $t \in R$ with $\psi_p(x) := t$, $\forall\ x \in R$. Thus, for any $a,b \in R$, we have

$$\begin{aligned} a\cdot b &= \varphi_p(ao_pb) + \psi_p(a+b) \\ &= \varphi_p(ao_pb) + t \\ &= \varphi_p((a\cdot b)o_p1) + t. \end{aligned}$$

Therefore, $\varphi_p(ao_pb) = \varphi_p((a\cdot b)o_p1)$. But, φ_p is one-to-one function and hence, $ao_pb = (a\cdot b)o_p1$. This completes the proof. □

THEOREM 12. The following properties are equivalent in any groupoid (R,o_p); $p \neq 0$.

(1) $\exists\ f: R \to R$; $f(ao_p(b\cdot c)) = (a\cdot b)o_pc$, $\forall\ a,b,c \in R$,

(2) $ao_pb = (a\cdot b)o_p1$ and $(a\cdot b)\cdot c = a\cdot(b\cdot c)$, $\forall\ a,b,c \in R$,

(3) $ao_p(b\cdot c) = (a\cdot b)o_pc$, $\forall\ a,b,c \in R$.

Proof. It is easy to show that (2) and (3) are equivalent and (1) follows from (3). Remains to show (1) $\Rightarrow$ (3). Putting $c = 1$ in (1), we have $f(ao_pb) = (a\cdot b)o_p1$ and therefore $(a\cdot b)o_pc = f(ao_p(b\cdot c)) = a\cdot(b\cdot c)o_p1$. Also, $a = 1$ gives $bo_pc = (b\cdot c)o_p1$ and thus $ao_p(b\cdot c) = a\cdot(b\cdot c)o_p1 = (a\cdot b)o_pc$. □

2. THE QUASIGROUPS $(R,+_p)$ and $(R,*_p)$

Let π be any projective plane coordinatized by the ternary ring (R,F). Fixing an element $p \neq 0$ in R, and using the ternary operation F, two binary operations, denoted by $+_p$ and $*_p$, are to be defined as follows:

$$\begin{aligned} a+_pb &= F(a,p,b), \\ a*_pb &= F(p,a,b),\ \forall\ a,b \in R. \end{aligned}$$

We may write $a+_1b = a+b$ and $a*_1b = a*b$.

First we prove the following sequence of lemmas concerning the system $(R,+_p)$.

LEMMA 5. $(R,+_p)$ is a quasigroup with left identity element 0.

Proof. Since, $a+_pb = F(a,p,b)$, the system $(R,+_p)$ is a groupoid (binary system). Now consider the equation $a+_px = b$, $a,b \in R$. In fact, $a+_px = b \Leftrightarrow F(a,p,x) = b \Leftrightarrow (a,b)I[p,x] \Leftrightarrow (0,x) = [0]\cap[(p)(a,b)]$. Hence the equation has a unique solution. Also, $y+_pa = b \Leftrightarrow F(y,p,a) = b \Leftrightarrow (y,b)I[p,a] \Leftrightarrow (y,b) = [0,b]\cap[p,a]$ and consequently the equation $y+_pa = b$ has a unique solution. Finally, for any $a \in R$, $0+_pa = F(0,p,a) = a$. This shows that 0 is the left identity of the groupoid $(R,+_p)$. □

By Lemma 5, we have proved the following result.

LEMMA 6. The following two laws hold for all $a,b,c \in R$:

(i) $a+_pb = a+_pc \Rightarrow b = c$,

(ii) $b+_pa = c+_pa \Rightarrow b = c$.

We remark that since $a+_p0 = F(a,p,0) = a.p$, then $a+_p0 = a \Leftrightarrow a = 0$ or $p = 1$. Consequently, unless $p = 1$, 0 is not the right identity of $(R,+_p)$. In fact, if $p \neq 1$, the system $(R,+_p)$ has no right identity; because if e is a right identity, then $a+_pe = a$, $\forall\, a \in R$, which implies $0+_pe = 0$, i.e., $e = 0$; but $1+_p0 = 1 \Rightarrow (1,1)I\,[1, 0] \Rightarrow p = 1$. Thus we have proved the following lemma.

LEMMA 7. Unless $p = 1$, $(R,+_p)$ is not a loop.

LEMMA 8. Unless $p = 1$, the operation $+_p$ is non-commutative in any ternary ring (R,F) associated with a projective plane π.

Proof. For $p \neq 1$, we get $0+_p1 = 1$ and $1+_p0 = p$. In case $p = 1$, it has been shown that, the loop $(R,+)$ is abelian in those planes where the first minor proposition of Pappus, P_1, is valid [7, p.25]. □

LEMMA 9. Unless $p = 1$, the operation $+_p$ is non-associative in any ternary ring (R,F) associated with a projective plane π.

Proof. Taking $a = 1$ and $b = c = 0$, we get $(1+_p0)+_p0 = p.p$ and $1+_p(0+_p0) = p$. But $p^2 \neq p$ (as $p \neq 1$). For the special case $p = 1$, the loop $(R,+)$ is a group; i.e., + is associative operation, if in the plane coordinatized by (R, F) the Reidemeister proposition for the (X, Y, (1)) - net is valid [6]. □

It is to be noted that $0+_p(b+_pc) = (0+_pb)+_pc$, holds for any b,c in R. While, $a+_p(0+_pc) = (a+_p0)+_pc \Leftrightarrow a+_pc = (a\cdot p)+_pc \Leftrightarrow a = a\cdot p \Leftrightarrow a = 0$ or $p = 1$

Now, for the system $(R,*_p)$, we prove the following lemmas.

LEMMA 10. $(R,*_p)$ is a quasigroup with left identity element 0.

Proof. Since, $a*_pb = F(p,a,b)$, $(R,*_p)$ is a binary system (a groupoid). Now, $a*_px = b \Rightarrow F(p,a,x) = b \Rightarrow (p,b)\ I\ [a,x] \Rightarrow (0,x) = [0] \cap [(a)\ (p,b)]$; thus x is unique in R. Also, $y*_pa = b \Rightarrow F(p,y,a) = b \Rightarrow (p,b)I[y,a] \Rightarrow (y) = L \cap [(p,b)\ (0,a)]$. Thus, y is unique in R. Since $0*_pb = F(p,0,b) = b$, for any $b \in R$, we conclude that 0 is the left identity of the quasigroup $(R,*_p)$. □

Immediately, we obtain the result in Lemma 11.

LEMMA 11. The following two laws hold for any a,b,c in R:

(1) $a*_pb = a*_pc \Rightarrow b = c$;

(2) $b*_pa = c*_pa \Rightarrow b = c$.

LEMMA 12. Unless $p = 1$, $(R,*_p)$ is not a loop.

Proof. By the preceding lemma, $(R,*_p)$ is a quasigroup with the left identity element 0. Now, since $a*_p0 = p\cdot a$, then $a*_p0 = a \Leftrightarrow p = 1$ or $a = 0$. Thus we conclude that, unless $p = 1$, 0 is not the right identity of $(R,*_p)$. In fact, $a*_pe = a$, $\forall\ a \in R$ implies, $0*_pe = 0$, i.e., $e = 0$. Hence, unless $p = 1$, the system $(R,*_p)$ has no right identity and hence $(R,*_p)$ is not a loop. □

LEMMA 13. Unless $p = 1$, the quasigroup $(R,*_p)$ is not abelian.

Proof. If $p \neq 1$, we establish the result by taking $a = 0$ and $b = 1$ and then we obtain $0*_p1 = 1$, while $1*_p0 = p$. In the case $p = 1$, the system $(R,*)$ is an abelian group if the second minor proposition of Pappus is valid in the projective plane coordinatized by R [2]. □

Now, the question presents itself is: are the two operations $+_p$ and $*_p$ dual ? The question is answered for the the special case p = 1 [2]. Moreover, it has been shown [4] that the two dual operations + and * are equal if, and only if, the little proposition of Desargues is valid (see theorem 6 in[4]). In fact, it is not known yet if $+_p$ and $*_p$, $p \neq 0,1$, are dual. However, the following interesting theorem gives a necessary and sufficient condition that the two operations $+_p$ and $*_p$, $p \neq 0,1$, are equal in any projective plane coordinatized by a ternary ring (R,F).

THEOREM 13. In any projective plane π, the proposition of Pappus holds in π if, and only if, in every coordinatizing (R,F), $a*_pb = a+_pb$, for any $a,b \in R$; $p \in R\backslash\{0, 1\}$.

Proof. Since the proof of Theorem 16 in [5] is true when we take the element a(or b) to be any element $p \neq 0,1$, we conclude this result. □

THEOREM 14. D_1 is equivalent to each of the following conditions each in every ternary ring (R,F):

(i) $a{\cdot}p = p{\cdot}a \Rightarrow a*_pb = a+_pb$, $\forall\, a,b \in R$; $p \in R\backslash\{0, 1\}$

(ii) $1*_pb = 1+_pb$, $\forall\, b \in R$; $p \in R\backslash\{0, 1\}$

Proof. Assume D_1 holds, then (R,F) is linear and hence, $a{\cdot}p = p{\cdot}a \Rightarrow a{\cdot}p+b = p{\cdot}a+b \Rightarrow F(a,p,b) = F(p,a,b)$, i.e., $a+_pb = a*_pb$. Thus (i) holds true. (i) $\Rightarrow$ (ii) is obvious as $1{\cdot}p = p{\cdot}1$. Remains to show (ii) implies the validity of D_1. Now, since $1+_pb = p*b$ and $1*_pb = p+b$ and by setting $a = p \neq 0, 1$, Theorem 6 in [4] gives D_1. □

The influence of the configurational proposition D_2, D_3 and finally $\bar{F}$ on the quasigroups $(R,+_p)$ and $(R,*_p)$ is explained in the following Theorems.

THEOREM 15. D_2 holds in a projective plane if, and only if, in every coordinatizing (R,F) $a*_pa = 0$, $\forall\, a \in R$ with $p \neq 0$ determined by $p + 1 = 0$.

Proof. Let (1,2,3), (1′,2′,3′) be two triangles, in π, generating an incomplete D_2-configuration. Set (X,Y,Q,E) as depicted in Fig. 5. Assume that $p+1 = 0$; $p \neq 1$ (this means that $\bar{F}$ is not valid in π). Now, for $a \in R\backslash\{0,1\}$(excluding the trivial cases), we may set 5 = (a) and hence we find 3 = (1), 2= (0,1), 1′ = (1,a), 4 = (0,a) and 6 = (x,0)I[1,1] $\Rightarrow x+1 = 0 \Rightarrow x = p$. Now, D_2-configuration is complete $\Leftrightarrow$ (5 4 6) $\Leftrightarrow$ (p,0)I[a,a] $\Leftrightarrow a*_pa = 0$.□

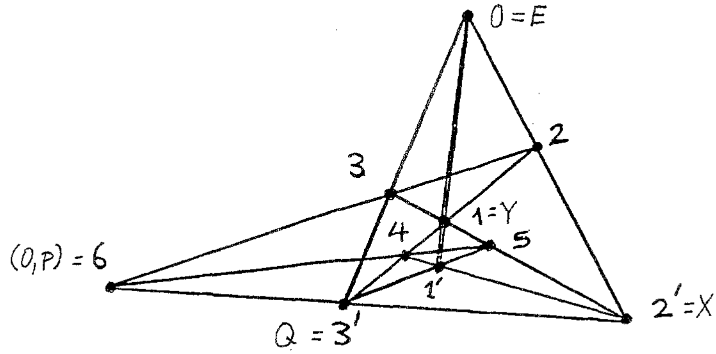

Fig. 5

THEOREM 16. D_3 holds in a projective plane if, and only if, in every coordinatizing (R, F), $(1 + 1)*_p(1 + 1) = 0$ with $p \neq 1$ determined by $p + 1 = 0$.

Proof. Let (1,2,3) and (1′,2′,3′) be two triangles, in π, generating an incomplete D_3-configuration (see Fig. 6). Choose the coordinate quadrangle as in the preceding theorem. Then, we get $1' = (1)$, $2' = (0,1)$, $3 = [1] \cap [1,1] = (1,1+1) = (1,a)$ and hence $2 = (0,a)$ and $1 = (a)$. Calculating the coordinates of 4, we find that $4 = (x,0)I[1,1] \Rightarrow x+1 = 0$ and hence $x = p$. Now, the configuration is complete $\Leftrightarrow (p,\bar{0})I[a,a] \Leftrightarrow a*_pa = 0$. □

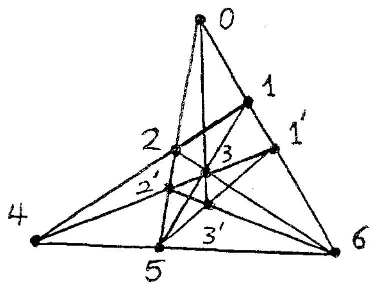

Fig. 6

It is to be noted that the proposition $\overline{F}$ follows, in π, only if $1 + 1 = 0$ is valid in every coordinatizing (R,F) and consequently D_2 and D_3 hold, in π, (see Theorem 2 in [4]). Therefore, the preceding two theorems give a necessary and sufficient conditions for the two propositions D_2 and D_3 to be valid in a non-Fano plane π (i.e., a plane where $\overline{F}$ is not valid).

Finally, we remark that the proof of Theorem 8 in [5] remains true when we take the element a or b to be any element $p \in R\backslash\{0\}$ and consequently we may state the following two results that explain the influence of the proposition $\overline{F}$ on the quasigroups $(R,+_p)$ and $(R,*_p)$, respectively.

THEOREM 17. $\overline{F}$ holds, in a projective plane π if, and only if, in every coordinatizing (R, F), $a +_p a.p = 0$, $\forall\, a \in R$.

COROLLARY 5. $\overline{F}$ holds, in a projective plane π, if, and only if, in every coordinatizing (R, F), $a *_p p.a = 0$, $\forall\, a \in R$.

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