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Functional equation problems in geometry

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Summary. Functional equations are playing an important role in geometry, especially in connection with invariants and invariant notions of different geometries. In the present note we present ten problems on *Functional Equations in Geometry*, hoping that such a collection might help to stimulate research in this specific discipline.

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1. Functional equations are playing an important role in geometry, especially in connection with invariants and invariant notions of different geometries (see, for instance, J. Aczél [1], J. Aczél and J. Dhombres [2], W. Benz [3], J. A. Lester [9], A. Schleiermacher [14]). In the present note we present ten problems on *Functional Equations in Geometry*, hoping that such a collection might help to stimulate research in this specific discipline.

2. A conditional functional equation. Let $\mathbb{L}$ be the set of all lines of $\mathbb{R}^3$. By $d\,(l_1, l_2)$ designate the distance of the lines $l_1, l_2 \in \mathbb{L}$.

Problem 1. *Prove or disprove that every mapping $f : \mathbb{L} \to \mathbb{L}$ satisfying*

$$d\big(f\,(l_1),\; f\,(l_2)\big) = 1,\;\; \textit{whenever}\; d\,(l_1, l_2) = 1 \tag{1}$$

holds true for $l_1, l_2 \in \mathbb{L}$, must be induced by a euclidean isometry of $\mathbb{R}^3$.

The best known result in this direction is the following theorem of June A. Lester [10].

Theorem. *If $f : \mathbb{L} \to \mathbb{L}$ is a bijection satisfying*

$$\forall_{l_1, l_2 \in \mathbb{L}}\; d\big(f\,(l_1), f\,(l_2)\big) = 1 \Leftrightarrow d\,(l_1, l_2) = 1,$$

then f is induced by a euclidean isometry of $\mathbb{R}^3$.

Let X be a pre-Hilbert space, i.e. a real vector space equipped with an inner product

$$\delta : X \times X \to \mathbb{R},\ \delta(x,y) =: xy,$$

satisfying $x^2 > 0$ for all $x \neq 0$ in X. If p, v are elements of X with $v \neq 0$, then

$$p + \mathbb{R}v := \{p + \lambda v \mid \lambda \in \mathbb{R}\}$$

is called a *line* of X. If $l_i = p_i + \mathbb{R}v_i$, $i = 1, 2$, are lines, define

$$d(l_1, l_2) := \inf_{\lambda_1,\lambda_2 \in \mathbb{R}} E(p_1 + \lambda_1 v_1,\ p_2 + \lambda_2 v_2)$$

with $E(x,y) := \sqrt{(x-y)^2}$ for $x, y \in X$. Assume $v_1^2 = 1 = v_2^2$, without loss of generality, and put $a := p_1 - p_2$. Then, obviously,

$$[d(l_1, l_2)]^2 = a^2 - (av_1)^2$$

for $v_2 \in \{v_1, -v_1\}$, and

$$[1 - (v_1v_2)^2] \cdot [d(l_1, l_2)]^2 = \begin{vmatrix} a^2 & av_1 & av_2 \\ av_1 & 1 & v_1v_2 \\ av_2 & v_1v_2 & 1 \end{vmatrix}$$

for $v_2 \notin \{v_1, -v_1\}$. Note $(v_1v_2)^2 \leq v_1^2v_2^2 = 1$ and, moreover, that $(v_1v_2)^2 = 1 = v_1^2v_2^2$ would imply $v_2 \in \{v_1, -v_1\}$.

Suppose now that X is a pre-Hilbert space of dimension at least 3. Of course, the dimension of X might be infinite. By $\mathbb{L}$ denote the set of all lines of X.

Problem 2. *Determine all* $f : \mathbb{L} \to \mathbb{L}$ *satisfying* (1) *for all* $l_1, l_2 \in \mathbb{L}$.

3. The isomorphism equation for geometries. Let X be a pre-Hilbert space of dimension at least 2 and let $O(X)$ be its orthogonal group. Suppose that e is a fixed element of X with $e^2 = 1$. Define

$$H := e^{\perp} = \{x \in X \mid xe = 0\}.$$

If $\varrho : H \times \mathbb{R} \to \mathbb{R}$ satisfies

> *For all* $h \in H$ *and* $\xi \in \mathbb{R}$ *there exists exactly one* $t = t(h, \xi) \in \mathbb{R}$ *with* $\varrho(h, t) = \xi$, $\qquad (*)$

then

$$\forall_{h \in H}\ \forall_{t,\tau \in \mathbb{R}}\ T_t\big(h + \varrho(h,\tau)\,e\big) := h + \varrho(h, \tau + t)\,e$$

defines a *translation group* of X with axis e. For translation groups and their characterization via the translation equation see [5]. In the case

$$\forall_{h \in H}\ \forall_{t \in \mathbb{R}}\ \varrho(h,t) = t \qquad (2)$$

we get the classical translations, and in the case

$$\forall_{h\in H}\ \forall_{t\in\mathbb{R}}\ \varrho\,(h,t)=\sinh t\cdot\sqrt{1+h^2} \tag{3}$$

the translations of hyperbolic geometry. The geometry (see [3])

$$\Gamma\,(T):=(X,G), \tag{4}$$

where T denotes a translation group $\{T_t \mid t\in\mathbb{R}\}$ and G the group generated by T and $O\,(X)$, will now be of interest. In the case (2), $\Gamma\,(T)$ is the *euclidean geometry*, and in the case (3) the *hyperbolic geometry* (see [5]).

The *isomorphism equation* for two geometries $\Gamma\,(T^i)$, $i=1,2$, is given (see [3]) by

$$\forall_{g\in G^1}\ \tau\,(g)\,\sigma=\sigma g. \tag{5}$$

A pair σ,τ is called a solution of (5) if σ is a bijection of X and $\tau: G^1\to G^2$ an isomorphism between the groups G^1 and G^2 such that (5) holds true.

Problem 3. *Given two functions ϱ_1 and ϱ_2 satisfying* $(*)$ *and given their corresponding groups T^1 and T^2. Find necessary and sufficient conditions on ϱ_1 and ϱ_2 such that the isomorphism equation has at least one solution, i.e. such that $\Gamma\,(T^1)$, $\Gamma\,(T^2)$ are isomorphic.*

For the geometries based on (2), (3), respectively, the isomorphism equation (5) has no solution.

There exist geometries (4) where G is generated by $O\,(X)$ and the translation group T with axis e such that

$$G=O\,(X)\cdot T\cdot O\,(X) \tag{6}$$

does not hold true (see [5]). However, in the cases (2), (3) equation (6) is satisfied. I think that it is important to classify all geometries (4) fulfilling (6). This is again a functional equations problem. If $t,s\in\mathbb{R}$ and $\omega\in O\,(X)$ are given, we are interested in $\alpha,\beta\in O\,(X)$ and $r\in\mathbb{R}$ such that

$$T_t\cdot\omega\cdot T_s=\alpha\cdot T_r\cdot\beta \tag{7}$$

holds true. $\alpha=\alpha\,(t,s,\omega)$, $\beta=\beta\,(t,s,\omega)$, $r=r\,(t,s,\omega)$ will be called a solution of (7).

Problem 4. *Find all geometries* (4) *where G is generated by $O\,(X)$ and T such that* (7) *is solvable for all $t,s\in\mathbb{R}$ and $\omega\in O\,(X)$.*

4. Lorentz transformations. Let X be a pre-Hilbert space of dimension at least 2 and $t\in X$ be a fixed element with $t^2=1$. If $x\in X$, there exist uniquely determined elements $\overline{x}\in H:=t^\perp$ and $x_0\in\mathbb{R}$ satisfying

$$x=\overline{x}+x_0t,$$

namely $\overline{x} = x - (xt)\, t$ and $x_0 = xt$. The Lorentz–Minkowski distance of $x, y \in X$ is defined by

$$l\,(x,y) = (\overline{x} - \overline{y})^2 - (x_0 - y_0)^2.$$

A mapping $f : X \to X$ is called a Lorentz transformation if, and only if,

$$\forall_{x,y \in X}\; l\,(x,y) = l\,\big(f\,(x),\; f\,(y)\big)$$

holds true. All Lorentz transformations can explicitly be determined ([6]) by means of Lorentz boosts. Up to euclidean translations they all must be linear mappings of X.

Problem 5. *Let $\varrho \neq 0$ be a fixed real number. Determine all $f : X \to X$ satisfying*

$$\forall_{x,y \in X}\; l\,(x,y) = \varrho \Rightarrow l\,\big(f\,(x),\; f\,(y)\big) = \varrho. \tag{8}$$

The best known result in the direction of this problem is the following theorem.

Theorem. *If $\dim X < \infty$, then exactly the Lorentz transformations f of X are the solutions of the conditional functional equation* (8)*.*

For $\varrho > 0$ this was proved by J. A. Lester, and for $\varrho < 0$ by W. Benz (see [4]).

5. Relativistic addition. Adding velocities p, q in mechanics, we get the usual vector addition $p+q$ as the result. A characterization of this phenomenon by means of functional equations is presented in [2]. Adding p and q in Special Relativity Theory, we get (see, for instance, [4])

$$p * q = \frac{p+q}{1+pq} + \frac{1}{1+\sqrt{1-p^2}}\,\frac{(pq)\,p - p^2 q}{1+pq}. \tag{9}$$

Problem 6. *For a pre-Hilbert space X of dimension at least one, define*

$$V := \{x \in X \mid x^2 < 1\}.$$

Observe $p * q \in V$ for all $p, q \in V$. Find a functional equations approach to the relativistic addition (9)*.*

We would like to mention a possible solution of Problem 6 (Benz [7]). The Weierstrass map $\mu : V \to X$,

$$\mu\,(x) := \frac{x}{\sqrt{1-x^2}},$$

is a bijection between V and X. Define the *separation* $S\,(p,q)$ of $p, q \in V$ by means of

$$S\,(p,q) := \frac{1-pq}{\sqrt{1-p^2}\,\sqrt{1-q^2}}.$$

Theorem. *Suppose* $\dim X \geq 2$. *Then* $f : V \times V \to V$ *is of the form*

$$f(p, q) = p * q$$

for all $p, q \in V$ *if, and only if,*

(i) $S(p, q) = S\big(f(x, p),\ f(x, q)\big)$,
(ii) $[f(p, 0)]^2 = p^2$,
(iii) $\mu\big(f(p, q)\big) - \mu(q) \in \mathbb{R}_{>0} \cdot p$

hold true for all $x, p, q \in V$, *where* $\mathbb{R}_{>0}$ *designates the set of all positive real numbers.*

In the case $\dim X = 1$, where $x \cdot y$ is defined as the usual product of $x, y \in \mathbb{R}$, a characterization of $f(p, q) = p * q$ is given ([7]) by (i) for all $x, p, q \in V$ and by

(ii)* $\forall_{p \in V}\ f(p, 0) = p$,
(iii)* $\exists_{p \in V}\ f(p, p) \neq 0$.

Another result of [7] is that two of the properties (i), (ii), (iii) do not imply the third one. It might also be mentioned that the properties (i), (ii) and

(iv) $\mu\big(f(p, q)\big) - \mu(q) \in \mathbb{R} \cdot p$

for all $x, p, q \in V$ do not characterize $f(p, q) = p * q$.

Of course, other solutions of Problem 6 would be appreciated.

6. Another conditional functional equation. Let R be a commutative and associative ring with identity element 1 such that $1 + 1$ is a unit in R. Designate $R \times R$ by R^2 and define

$$F(a, b, c) := \frac{1}{2} \begin{vmatrix} a_1 & a_2 & 1 \\ b_1 & b_2 & 1 \\ c_1 & c_2 & 1 \end{vmatrix}$$

for the elements $a = (a_1, a_2)$, $b = (b_1, b_2)$, $c = (c_1, c_2)$ of R^2.

Problem 7. *Determine all functions* $\varphi, \psi : R^2 \to R$ *such that*

$$\big[F\big(f(a),\ f(b),\ f(c)\big)\big]^2 = 1$$

holds true for all $a, b, c \in R^2$ *satisfying*

$$\big[F(a, b, c)\big]^2 = 1,$$

where we put $f(x) := \big(\varphi(x_1, x_2),\ \psi(x_1, x_2)\big)$ *for* $x = (x_1, x_2) \in R^2$.

The solution of this problem in the case $R := \mathbb{R}$ is the following theorem of G. Martin (see [3], section 5.3.1).

Theorem. *Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a mapping satisfying*

$$\forall_{a,b,c\in\mathbb{R}^2} \quad \triangle(a,b,c) = 1 \Rightarrow \triangle\big(f\,(a),\; f\,(b),\; f\,(c)\big) = 1 \tag{10}$$

where $\triangle(a,b,c)$ denotes the area of the triangle with vertices a, b, c. Then f is an equiaffine mapping of $\mathbb{R}^2$.

7. Generalizations of the theorem of Beckman and Quarles. F. Radó [12], [13] has presented extensions of the theorem of Beckman and Quarles (see [4], chapter 1) to the case of Galois fields.

Problem 8. *Find and study a general version of the theorem of Beckman and Quarles for the field case.*

8. Area 1 preserving mappings. June Lester proved ([11], see also [3], section 5.1.2) that a mapping $f : \mathbb{R}^n \to \mathbb{R}^n,\ n \geq 3$, satisfying (10) for $\mathbb{R}^n$ must be a euclidean isometry.

Problem 9. *Generalize this result to the case of an arbitrary pre-Hilbert space X of dimension at least 3.*

Let X be a pre-Hilbert space X with $\dim X \geq 3$ and designate by $\mathbb{L}$ the set of all lines of X. Wen-ling Huang ([8], see also [3], section 6.4.2) proved that a mapping $\pi : \mathbb{L} \to \mathbb{L}$, $\dim X < \infty$, satisfying

Whenever $a, b, c \in \mathbb{L}$ are lines which are sides of a triangle of area 1,
then also $\pi\,(a),\ \pi\,(b),\ \pi\,(c)$ are sides of a triangle of area 1

must be induced by a euclidean isometry of X.

Problem 10. *Generalize the theorem of Wen-ling Huang to the infinite-dimensional case.*

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