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Results in Mathematics

Closure Spaces of Finite Type

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Dedicated to Heinrich Wefelscheid on the occasion of his 70th birthday

Abstract. As well known in a closure space $(M, \mathfrak{D})$ satisfying the exchange axiom and the finiteness condition we can complete each independent subset of a generating set of M to a basis of M (Theorem A) and any two bases have the same cardinality (Theorem B) (cf. [1,3,4,7]). In this paper we consider closure spaces *of finite type* which need not satisfy the finiteness condition but a weaker condition (cf. Theorem 3.5). We prove Theorems A and B for a closure space of finite type satisfying a stronger exchange axiom. An example is given satisfying this strong exchange axiom, but not Theorems A and B.

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1. Introduction

We consider a set M and define a set $\mathfrak{D} \subset \mathfrak{P}(M)$ of subsets of M such that $\mathfrak{D}$ is a complete lattice, i.e., $\mathfrak{D}$ is closed with respect to intersections. We call the elements of $\mathfrak{D}$ *subspaces* of M.

For every subset $X \subset M$ we define the following *closure operation*:

$$^{-} : \mathfrak{P}(M) \to \mathfrak{D} : \ X \mapsto \overline{X} := \bigcap_{\substack{U \in \mathfrak{D} \\ X \subset U}} U$$

and call the pair $(M, \mathfrak{D})$ a *closure space*. A subset $X \subset M$ is *independent*, if $x \notin \overline{X \setminus \{x\}}$ for each $x \in X$, and a *basis* of a subspace U, if X is independent and $\overline{X} = U$.

Birkhäuser

We consider the following properties:

FC Finiteness Condition For each subset $T \subset M$ and $x \in \overline{T}$ there exists a finite subset $R \subset T$ with $x \in \overline{R}$.

FA Minimal Condition For each subset $T \subset M$ and $x \in \overline{T}$ there exists a minimal subset $R \subset T$ with $x \in \overline{R}$.

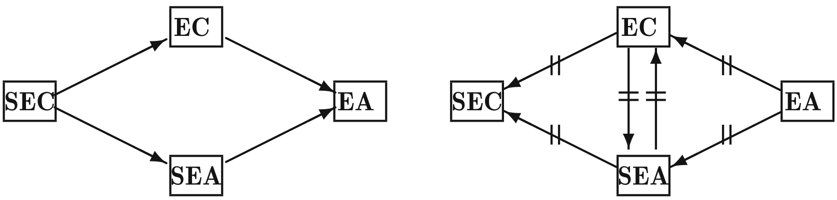
EA Exchange Axiom For each independent subset $S \subset M$, $y \in M$ and $x \in \overline{S \cup \{y\}} \setminus \overline{S}$ it holds $y \in \overline{S \cup \{x\}}$.

EC Exchange Condition For $S \subset M$, $y \in M$ and $x \in \overline{S \cup \{y\}} \setminus \overline{S}$ it holds $y \in \overline{S \cup \{x\}}$.

SEA Strong Exchange Axiom For independent subsets $S, Y \subset M$ with $Y \cap \overline{S} = \emptyset$ and $x \in \overline{S \cup Y} \setminus \overline{S}$ there exists $y \in Y$ with $y \in \overline{S \cup (Y \setminus \{y\}) \cup \{x\}}$.

SEC Strong Exchange Condition For subsets $S, Y \subset M$ with $Y \cap \overline{S} = \emptyset$ and $x \in \overline{S \cup Y} \setminus \overline{S}$ there exists $y \in Y$ with $y \in \overline{S \cup (Y \setminus \{y\}) \cup \{x\}}$.

We recall from [6] Theorems (3.2) and (3.5):



If **FA** holds, then **EA, EC, SEA, SEC** are equivalent (cf. [6], Theorem 3.4).

Theorem A. *For each subset $N \subset M$ and each independent subset $R \subset N$ there is an independent subset $B \subset N$ with $R \subset B$ and $\overline{B} = \overline{N}$, i.e., B is a basis of the subspace $\overline{N}$.*

Theorem B. *Any two bases of a subspace have the same cardinality.*

We recall (cf. [6], Theorem 2.2) that Theorem A implies **EA** and **FA**, and therefore also **SEC** and the following property:

S2 If $T \subset M$ is independent then for $Y \subset T$ it holds $\overline{T \setminus Y} = \bigcap_{y \in Y} \overline{T \setminus \{y\}}$.

For $T \subset M$, $Y \subset T$ and $D := \bigcap_{y \in Y} \overline{T \setminus \{y\}}$ we define the conditions

S If for all $y \in Y$ it holds $y \notin \overline{T \setminus \{y\}}$ then $\overline{T \setminus Y} = D$.

S1 If $T \setminus Y$ is independent and for all $y \in Y$ it holds $y \notin \overline{T \setminus \{y\}}$ then $\overline{T \setminus Y} = D$.

Lemma 1.1. (1) **S** $\Rightarrow$ **S1** $\Rightarrow$ **S2** (2) **S** *&* **EC** $\Rightarrow$ **SEC**. (3) **SEA** $\Rightarrow$ **S1**.

Proof. (2) Let $T := S \cup Y$ and $x \in \overline{T} \setminus \overline{S}$. If there is a $y \in Y$ with $y \in \overline{S \cup (Y \setminus \{y\})}$ then **SEC** is true. If for all $y \in Y$ it holds $y \notin \overline{S \cup (Y \setminus \{y\})}$ then by **S** there is a $y \in Y$ with $x \notin \overline{S \cup (Y \setminus \{y\})}$. By **EC** it follows $y \in \overline{S \cup (Y \setminus \{y\}) \cup \{x\}}$.

(3) With $S := T \setminus Y$ obviously $\overline{S} \subset D$ and $D \cap Y = \emptyset$, therefore $Y \cap \overline{S} = \emptyset$. For $x \in \overline{S \cup Y} \setminus \overline{S}$ by **SEA** there is a $y \in Y$ with $y \in \overline{S \cup (Y \setminus \{y\}) \cup \{x\}}$. As $y \notin \overline{T \setminus \{y\}}$ it follows $x \notin \overline{T \setminus \{y\}}$. Therefore $x \notin D$ and $D \subset \overline{S}$. □

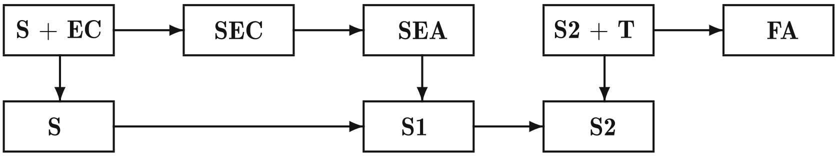
One half of Theorem A is the condition

T For each subset $N \subset M$ there is an independent subset $B \subset N$ with $\overline{B} = \overline{N}$.

In [6] Theorem 2.2 (4) the following lemma was proved:

Lemma 1.2. *Let $(M, \mathfrak{D})$ be a closure space with* **S2** *and* **T**, *then* **FA** *is true.*

We summarize:



2. Semi Independence

Let $(M, \mathfrak{D})$ be a closure space and $X \subset M$. Then X is called **semi independent** if each finite subset of X is independent.

Each independent set is semi independent and finite semi independent sets are independent.

Obviously the set $\mathfrak{S}$ of all semi independent subsets of M is a closure system.

Lemma 2.1. (1) *Let $\mathfrak{K}$ be a nonempty chain of $\mathfrak{S}$. Then $T := \bigcup_{K \in \mathfrak{K}} K \in \mathfrak{S}$, i.e., $\mathfrak{S}$ is inductive.*

(2) *For each subset $N \subset M$ and each semi independent subset $R \subset N$, the set $\{X \subset N \,:\, R \subset X \text{ and } X \text{ semi independent}\}$ has a maximal element B.*

Proof. (1). Let $R \subset T$ be finite. For each $x \in R$ there is a $K_x \in \mathfrak{K}$ with $x \in K_x$ and $L = \bigcup_{x \in R} K_x \in \mathfrak{K}$, hence R is independent and T is semi independent.

(2) follows by (1) and Zorn's lemma. □

We remark that the set B in Lemma 2.1(2) is not necessarily a generating set of $\overline{N}$.

Now let $(M, \mathfrak{D})$ be a closure space satisfying the exchange axiom **EA**.

Lemma 2.2. *Let $S \subset M$ be semi independent. Then:*

(1) *For each $x \in M \setminus \overline{S}$ the set $S \cup \{x\}$ is semi independent.*

(2) *Let $x \in \overline{S}$. If $S \cup \{x\}$ not semi independent, then there exists exactly one minimal finite subset $R \subset S$ with $x \in \overline{R}$*

Proof. (1). By Kreuzer and Sörensen [6] Lemma 1.1(3), for each independent subset $R \subset S$ and $x \notin \overline{R}$ the set $R \cup \{x\}$ is independent.

(2). As $S \cup \{x\}$ is not semi independent there is a finite dependent subset of $S \cup \{x\}$. As each finite subset of S is independent there is a finite subset $R \subset S$ such that $R \cup \{x\}$ is dependent. As R is independent from **EA** (cf. [6], Lemma 1.1(2)) it follows $x \in \overline{R}$. As R is finite there is a minimal subset $R_1 \subset R$ with $x \in \overline{R_1}$.

Let $R_1, R_2 \subset S$ be minimal finite subsets with $x \in \overline{R_1} \cap \overline{R_2} \setminus \overline{\emptyset}$ and $r \in R_1$. As $x \notin \overline{R_1 \setminus \{r\}}$ by **EA** it follows $r \in \overline{(R_1 \setminus \{r\}) \cup \{x\}} \subset \overline{(R_1 \setminus \{r\}) \cup R_2}$. As $R_1 \cup R_2$ is finite and therefore independent it follows $r \notin \overline{(R_1 \cup R_2) \setminus \{r\}}$ and therefore $(R_1 \cup R_2) \setminus \{r\} \subsetneq (R_1 \setminus \{r\}) \cup R_2$ and $r \in R_2$. □

Theorem 2.3. *For each subset $N \subset M$ and each semi independent subset $R \subset N$, there is a maximal semi independent set $S \subset N$ with $R \subset S$ and $\overline{S} = \overline{N}$.*

Proof. By Lemma 2.1(2) there exists a maximal element S and by Lemma 2.2 (1) we have $\overline{S} = \overline{N}$. □

Remark. In general a semi independent set N is not maximal independent in $\overline{N}$: let $\mathfrak{D}$ be the system of closed subsets of $\mathbb{R}$ and $N = \mathbb{Q}$, then $\overline{N} = \mathbb{R}$.

Lemma 2.4. *Let $A \subset M$ be finite and independent and $B \subset \overline{A}$ semi independent, then $|B| \leq |A|$. If B is maximal semi independent in $\overline{A}$, then $|B| = |A|$.*

Proof. By Theorem 2.3 there is a maximal semi independent subset $C \subset \overline{A}$ with $B \subset C$ and $\overline{A} = \overline{C}$.

Because of **EA** Steinitz' exchange theorem is valid in $(M, \mathfrak{D})$. Therefore in C there is a basis A' of $\overline{A}$ (cf. [1,7]) and further as A is finite, $|A'| = |A|$. As A' is maximal independent in C and C is semi independent it follows $A' = C$. □

Theorem 2.5. *Any two maximal semi independent subsets of a subspace have the same cardinality.*

Proof. Let $D \in \mathfrak{D}$ and A, B maximal semi independent subsets in D. Then by Theorem 2.3 $D = \overline{A} = \overline{B}$. If B is finite then by Lemma 2.4 $|A| = |B|$.

Let A, B be infinite. For $a \in A$ by Lemma 2.2(2) there is exactly one minimal finite subset $B_a \subset B$ with $a \in \overline{B_a}$. By Lemma 2.4 it holds $|\overline{B_a} \cap A| \leq |B_a|$.

By Karzel et al. [5, p. 191] for each infinite set C and $\mathfrak{B} \subset \{X \in \mathfrak{P}(C) : |X| \in \mathbb{N}_0\}$ with $\bigcup_{R \in \mathfrak{B}} R = C$ it follows $|\mathfrak{B}| = |C|$. Therefore if $\mathfrak{B} := \{B_a : a \in A\}$ then from $\bigcup_{B_a \in \mathfrak{B}} (\overline{B_a} \cap A) = A$ and $\bigcup_{B_a \in \mathfrak{B}} B_a \subset B$ it follows $|\mathfrak{B}| = |A|$ and $|\mathfrak{B}| \leq |B|$. □

3. Near Independence

Let $(M, \mathfrak{D})$ be a closure space. A subset $S \subset M$ is said to be **nearly independent** if there is an $n \in \mathbb{N}_0$ such that

$$\text{For each } X \subset S \text{ with } X \subset \overline{S \backslash X} \text{ it holds } |X| \leq n.$$

If we need to make use of the number n then the nearly independent set S sometimes will be called n-**independent**. We remark that a subset N of an n-independent set S my be m-independent for a smaller $m \leq n$.

Lemma 3.1. *Let $N \subset M$ be nearly independent and $T \subset N$. Then there is a maximal (and finite) subset $Z \subset T$ with $Z \subset \overline{N \backslash Z}$*

Lemma 3.2. *Let $N \subset M$ be nearly independent then there is an independent subset $B \subset N$ with $\overline{B} = \overline{N}$*

Proof. Let $N_0 := N$. If N_i is dependent let $x_{i+1} \in N_i$ with $x_{i+1} \in \overline{N_i \backslash \{x_{i+1}\}}$ and $N_{i+1} := N_i \backslash \{x_{i+1}\}$. If N is n-independent with $n \in \mathbb{N}_0$ then there is $0 \leq i \leq n$ such that N_i is independent. □

Theorem 3.3. *Let $(M, \mathfrak{D})$ be a closure space with* **SEA**. *Let $R \subset S \subset M$, R independent and S nearly independent. Then there is an independent set $B \subset S$ with $R \subset B$ and $\overline{B} = \overline{S}$.*

Proof. Let $Z \subset S \backslash R$ be maximal with $Z \subset \overline{S \backslash Z}$ (cf. Lemma 3.2), $V := S \backslash (Z \cup R)$ and $B := S \backslash Z = R \mathbin{\dot{\cup}} V$. As Z is maximal for all $y \in V$ it holds $y \notin \overline{S \backslash (Z \cup \{y\})} = \overline{R \cup (V \backslash \{y\})}$ and V is independent. By **SEA** for all $x \in R$ it follows $x \notin \overline{(R \backslash \{x\}) \cup V}$. Therefore B is independent. □

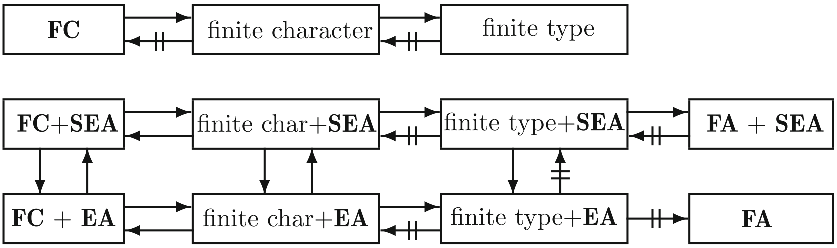
We recall that a closure space has **finite character** ([8, p. 7]) if each semi independent set is independent. This condition is weaker than the finiteness condition **FC** (cf. example in [2, page 160]). But if in addition the exchange axiom **EA** is valid, then each closure space with finite character satisfies the finiteness condition (cf. [2, page 169]).

We will say that a closure space is of **finite type** if each semi independent set is nearly independent. For a closure space to be of finite type in general is a weaker condition than to have finite character.

Example 3.4. The closure space $(M, \mathfrak{D})$ in example B in [6] with an infinite set M, $a \in M$ and $\mathfrak{D} := \{D \subset M : |D| \in \mathbb{N}_0\} \cup \{D \subset M : a \in D\}$ is not of finite character but of finite type as all semi independent subsets are 1-independent. $(M, \mathfrak{D})$ satisfies **EC** but not **SEA**. Therefore Theorem A is not true in $(M, \mathfrak{D})$: Let $N \subset M$ be dependent and $R \subset N$ be finite with $a \in R$, then $\overline{N} = N$ and R is independent. The set $B := N \backslash \{a\}$ is the only independent subset of N with $\overline{B} = N$, but $R \not\subset B$.

We summarize:

Theorem 3.5.



Proof. If **FC** or **FA** holds, by Theorem (3.4) of [6], **EA** and **SEA** are equivalent. For a closure space of finite type satisfying **SEA**, by the following Lemma 3.7, **FA** holds. The example of Sect. 4 shows that a closure space with **SEA** and **FA** in general does not satisfy Theorem B, hence by the following theorem 3.6 the closure space cannot be of finite type.

Example C of [6] is a closure space of finite type (since all semi independent sets are 1-independent) with **SEA**, not satisfying **FC**, hence a closure space of finite type satisfying **SEA** need not have finite character. □

Theorem 3.6. *Let* $(M, \mathfrak{D})$ *be a closure space of finite type satisfying the exchange axiom* **EA**. *Then:*

(1) (Condition **T**) *For each* $N \subset M$ *there is an independent subset* $B \subset N$ *with* $\overline{B} = \overline{N}$.

(2) (Theorem **B**) *Every two bases of a subspace have the same cardinality.*

Proof. (1). By Theorem 2.3 there is a semi independent $A \subset N$ with $\overline{A} = \overline{N}$. By Lemma 3.2 there is an independent subset $B \subset A$ with $\overline{B} = \overline{A}$.

(2) Let $D \in \mathfrak{D}$ and B_1, B_2 bases of D. If B_1 is finite then by Lemma 2.4 we have $|B_1| = |B_2|$. Let B_1, B_2 be infinite. By Theorem 2.3 there are maximal semi independent sets A_1, A_2 with $B \subset A_i \subset D$. By Theorem 2.5 it holds $|A_1| = |A_2|$. As $A_i \backslash B_i \subset D = \overline{B_i} = \overline{A_i \backslash (A_i \backslash B_i)}$ and A_i is semi independent and therefore nearly independent it follows $|A_i \backslash B_i| \in \mathbb{N}_0$ and $|B_i| = |A_i|$. □

We remark that by Theorem 2.3 we can complete an independent set R of a subspace U to a maximal semi independent set N and by Theorem 3.6(1) N contains a basis B, but in general we have not $R \subset B$. Now we assume in addition **SEA**.

Lemma 3.7. *Let* $(M, \mathfrak{D})$ *be a closure space of finite type with* **SEA**, *then* $(M, \mathfrak{D})$ *satisfies the minimal condition* **FA**.

Proof. As **SEA** includes **EA**, by Theorem 3.6 $(M, \mathfrak{D})$ satisfies the condition **T**. By Lemma 1.1 from **SEA** follows **S1** and by Lemma 1.2 the condition **FA** is true. □

Theorem 3.8. *Let* $(M, \mathfrak{D})$ *be a closure space of finite type with* **SEA***, then:* (*Theorem* **A**). *For each subset* $N \subset M$ *and for each independent subset* $R \subset N$*, there exists a basis* B *of* $\overline{N}$ *with* $R \subset B \subset N$.

Proof. By Lemma 2.1(2) there exists a semi independent subset B with $R \subset B \subset N$ which is maximal in N and by Theorem 2.3, $\overline{B} = \overline{N}$. By assumption B is nearly independent and the proposition follows by Theorem 3.3 □

4. Example

Let M be a set and $\mathfrak{F} \subset \mathfrak{P}(M)$. $\mathfrak{F}$ is called a **c-system** on M, if for all $D \in \mathfrak{F}$ and $C \subset D$ it holds $C \in \mathfrak{F}$.

For $\mathfrak{F}^0 \subset \mathfrak{P}(M)$ the set $\mathfrak{F} := \{D \subset M : \exists\ F \in \mathfrak{F}^0 \text{ with } D \subset F\}$ is a c-system, **determined** by $\mathfrak{F}^0$.

Lemma 4.1. *For each family* $(\mathfrak{F}_\lambda)_{\lambda \in \Lambda}$ *of c-systems on* M *the sets* $\bigcap_\lambda \mathfrak{F}_\lambda$ *and* $\bigcup_\lambda \mathfrak{F}_\lambda$ *are c-systems.*

Lemma 4.2. *Let* $\mathfrak{F}$ *be a c-system on* M.

(1) $M \in \mathfrak{F}$ *if and only if* $\mathfrak{F} = \mathfrak{P}(M)$.
(2) $\mathfrak{F} \cup \{M\}$ *is a closure system with the corresponding closure operator*

$$\overline{S} = \begin{cases} S & \text{if } S \in \mathfrak{F} \cup \{M\} \\ M & \text{if } S \notin \mathfrak{F} \end{cases}$$

(3) *Each* $T \in \mathfrak{F} \backslash \{M\}$ *is independent.*
(4) $T \subset M$ *is independent if and only if for each* $y \in T$ *it holds* $T \backslash \{y\} \in \mathfrak{F}$.

Lemma 4.3. *Let* $\mathfrak{F}$ *be a c-system on* M*. In the closure space* $(M, \mathfrak{F} \cup \{M\})$ *the conditions* **EA**, **EC** *and* $(*)$ *are equivalent.*

$(*)$ For all $S \in \mathfrak{F}$ and $y \in M$ with $S \cup \{y\} \notin \mathfrak{F}$ the set S is a maximal element in $\mathfrak{F}$.

Proof. **EA** $\Rightarrow (*)$. S is independent as $S \in \mathfrak{F}$, and $\overline{S \cup \{y\}} = M$ as $\overline{S \cup \{y\}} \notin \mathfrak{F}$. For $x \notin S = \overline{S}$ it follows $x \in \overline{S \cup \{y\}} \backslash \overline{S}$ and by **EA** $y \in \overline{S \cup \{x\}}$, hence $\overline{S \cup \{x\}} = M$.

$(*) \Rightarrow$ **EC**. Let $S \subset M$, $y \in M$ and $x \in \overline{S \cup \{y\}} \backslash \overline{S}$, then $M \neq \overline{S} = S \in \mathfrak{F}$. If $x \neq y$ then $x \notin S \cup \{y\} \neq \overline{S \cup \{y\}}$ and $\overline{S \cup \{y\}} \notin \mathfrak{F}$. By $(*)$ the set S is maximal and therefore $\overline{S \cup \{x\}} = M$. □

Lemma 4.4. *Let* $(M, \mathfrak{F} \cup \{M\})$ *be a closure space with* **EA** *and* $B \subset M$*. Then the following conditions are equivalent:*

(1) *For each* $y \in B$ *the set* $B \backslash \{y\}$ *is maximal in* $\mathfrak{F}$.
(2) $B \notin \mathfrak{F}$ *and* B *is a basis.*

Proof. (1) ⇒ (2). By Lemma 4.2(4) the set B is independent. As $B\backslash\{y\}$ is maximal it follows $B \notin \mathfrak{F}$ and $\overline{B} = M$.

(2) ⇒ (1). By Lemma 4.2(4) for each $y \in B$ it holds $B\backslash\{y\} \in \mathfrak{F}$. Let $x \in M\backslash B$ then $x \in \overline{B}\backslash\overline{B\backslash\{y\}}$. By **EA** it follows $y \in \overline{(B\backslash\{y\}) \cup \{x\}} = M$ and $(B\backslash\{y\}) \cup \{x\} \notin \mathfrak{F}$. □

In the following we will give examples of c-systems on a set M. Let Q_1, Q_2 be disjoint infinite sets, $M := Q_1 \cup Q_2$ and for $i \in \{1,2\}$ and $S \subset M$ define

$$\delta_i(S) := |Q_i \cap S| \quad \text{and} \quad \mu_i(S) := |Q_i\backslash S|.$$

Lemma 4.5. *Let $i, j \in \{1,2\}$ with $i \neq j$, $R \subset S \subset M$ and $y \in Q_j\backslash S$.*

(1) $\delta_i(R) \leq \delta_i(S)$ *and* $\mu_i(R) \geq \mu_i(S)$.
(2) $\delta_j(S \cup \{y\}) = \delta_j(S) + 1$, $\mu_j(S \cup \{y\}) = \mu_j(S) - 1$, $\delta_i(S \cup \{y\}) = \delta_i(S)$ *and* $\mu_i(S \cup \{y\}) = = \mu_i(S)$.

For $i, j \in \{1,2\}$ with $i \neq j$ let

$$\begin{aligned}
\mathfrak{A}_i^0 &:= \{D \subset M : \delta_j(D) = \mu_i(D) \in \mathbb{N}_0\},\\
\mathfrak{B}_i &:= \{D \subset M : \delta_j(D) \in \mathbb{N}_0,\, \mu_i(D) \notin \mathbb{N}_0\},\\
\mathfrak{A}_i &:= \{D \subset M : \delta_j(D) \leq \mu_i(D) \in \mathbb{N}_0\} \,\cup\, \mathfrak{B}_i,\\
\mathfrak{C}_i^0 &:= \{D \subset M : \delta_j(D),\, \mu_i(D) \notin \mathbb{N}_0\},\\
\mathfrak{C}_i &:= \{D \subset M : \mu_i(D) \notin \mathbb{N}_0\}, \quad \text{and}\\
\mathfrak{D}_i &:= \mathfrak{A}_i \,\cup\, \mathfrak{C}_i^0.
\end{aligned}$$

From Lemma 4.5 it follows

Lemma 4.6. *Let $i, j \in \{1,2\}$ with $i \neq j$. Then*

(1) $\mathfrak{B}_i$ *and* $\mathfrak{C}_i$ *are c-systems.*
(2) *The set* $\mathfrak{A}_i^0$ *determines the c-system* $\mathfrak{A}_i$.
(3) *The set* $\mathfrak{C}_i^0$ *determines the c-system* $\mathfrak{C}_i$ *and* $\mathfrak{A}_i \cap \mathfrak{C}_i = \mathfrak{B}_i$.
(4) $\mathfrak{D}_i = \mathfrak{A}_i \cup \mathfrak{C}_i^0$ *is a c-system with* $\mathfrak{A}_i = \{D \in \mathfrak{D}_i : \delta_j(D) \in \mathbb{N}_0\}$.
(5) $\mathfrak{A}_i^0 \subset \mathfrak{A}_i\backslash\mathfrak{B}_i \subset \{D \subset M : \delta_j(D), \mu_i(D) \in \mathbb{N}_0\} \subset \mathfrak{C}_j$.
(6) $\mathfrak{A}_i^0 \cap \mathfrak{A}_j = \emptyset$.
(7) $\mathfrak{D}_1 \cap \mathfrak{D}_2 = \mathfrak{A}_1 \cup \mathfrak{A}_2 \cup (\mathfrak{C}_1^0 \cap \mathfrak{C}_2^0)$.
(8) $Q_i \in \mathfrak{A}_i^0$.

Proof. (5) Let $D \subset M$ with $\mu_i(D), \delta_j(D) \in \mathbb{N}_0$. From $|Q_i\backslash D| \in \mathbb{N}_0$ and $|Q_i| \notin \mathbb{N}_0$ it follows $\delta_i(D) = |Q_i \cap D| \notin \mathbb{N}_0$. From $|Q_j \cap D| \in \mathbb{N}_0$ and $|Q_j| \notin \mathbb{N}_0$ it follows $\mu_j(D) = |Q_j\backslash D| \notin \mathbb{N}_0$. □

Lemma 4.7. *Let $i \in \{1,2\}$, $S \in \mathfrak{D}_i$ and $y \in M\backslash S$. Then*

(1) *If* $S \in \mathfrak{A}_i$ *and* $S \cup \{y\} \in \mathfrak{D}_i$ *then* $S \cup \{y\} \in \mathfrak{A}_i$.
(2) $S \in \mathfrak{A}_i^0 \iff S \cup \{y\} \notin \mathfrak{D}_i$.
(3) *If* $S \in \mathfrak{A}_i^0$ *and* $s \in S$ *then* $(S\backslash\{s\}) \cup \{y\} \in \mathfrak{A}_i^0$.

Proof. Let $j \in \{1,2\}\backslash\{i\}$. (1) As $\delta_j(S) \in \mathbb{N}_0$ from Lemma 4.5(2) it follows $\delta_j(S \cup \{y\}) \in \mathbb{N}_0$ and $S \cup \{y\} \in \mathfrak{A}_i$ by Lemma 4.6(4).

(2) If $S \in \mathfrak{A}_i^0$ then $\delta_j(S) = \mu_i(S) \in \mathbb{N}_0$ and by Lemma 4.5(2) it follows $\delta_j(S \cup \{y\}) = \mu_i(S \cup \{y\}) + 1$.

If $S \cup \{y\} \notin \mathfrak{D}_i$ then $\delta_j(S \cup \{y\}) > \mu_i(S \cup \{y\}) \in \mathbb{N}_0$. By Lemma 4.5(2) it follows $\delta_j(S) + 1 > \mu_i(S) \in \mathbb{N}_0$ and therefore $S \in \mathfrak{A}_i$ and $\delta_j(S) \leq \mu_i(S)$.

(3) As $S\backslash\{s\} \in \mathfrak{A}_i\backslash\mathfrak{A}_i^0$ by (2) it follows $(S\backslash\{s\}) \cup \{y\} \in \mathfrak{D}_i$ and as $S \cup \{y\} \notin \mathfrak{D}_i$ by (2) again $(S\backslash\{s\}) \cup \{y\} \in \mathfrak{A}_i^0$. □

From Lemma 4.7 it follows

Lemma 4.8. (1) $\mathfrak{A}_i^0$ *is the set of all maximal elements of* $\mathfrak{A}_i$.
(2) $\mathfrak{A}_1^0 \cup \mathfrak{A}_2^0$ *is the set of all maximal elements of* $\mathfrak{D}_1 \cap \mathfrak{D}_2$.

In the following let $\mathfrak{F} := \mathfrak{A}_1 \cup \mathfrak{A}_2$ or $\mathfrak{F} := \mathfrak{D}_1 \cap \mathfrak{D}_2$.

By combining the Lemmata 4.3, 4.4, 4.7 and 4.8 we get the following results.

Lemma 4.9. EC *holds in* $(M, \mathfrak{F} \cup \{M\})$.

Lemma 4.10. *All bases of* $(M, \mathfrak{F} \cup \{M\})$ *are of the form* $S \cup \{y\}$ *with* $S \in \mathfrak{A}_1^0 \cup \mathfrak{A}_2^0$ *and* $y \in M\backslash S$.

Lemma 4.11. FC *does not hold in* $(M, \mathfrak{F} \cup \{M\})$.

Proof. With $q \in Q_2$ the set $Q_1 \cup \{q\}$ is a basis of M. By Lemma 4.1,(1) each proper subset $R \subsetneq Q_1 \cup \{q\}$ lies in $\mathfrak{F}$. Therefore $\overline{R} = R \neq M$ and $\overline{R} \cap (Q_2\backslash\{q\}) = \emptyset$. □

Lemma 4.12. (1) (Condition **T**) *In the closure space* $(M, (\mathfrak{D}_1 \cap \mathfrak{D}_2) \cup \{M\})$ *each generating set of* M *contains a basis of* M.
(2) **FA** *holds in* $(M, (\mathfrak{D}_1 \cap \mathfrak{D}_2) \cup \{M\})$.

Proof. (1) Let $N \subsetneq M$ with $\overline{N} = M$, then $N \notin \mathfrak{D}_1 \cap \mathfrak{D}_2$. If $N \notin \mathfrak{D}_1$ then $\delta_2(N) > \mu_1(N) \in \mathbb{N}_0$. Take a subset $R \subset Q_2 \cap N$ with $|R| = \mu_1(N)$. For $S := (Q_1 \cap N) \cup R$ we get $\delta_2(S) = \delta_2(R) = \mu_1(N) = \mu_1(S)$ and therefore $S \in \mathfrak{A}_1^0$. With $y \in (Q_2 \cap N)\backslash R$ the set $S \cup \{y\}$ is a basis by Lemma 4.10. □

Lemma 4.13. *For* $i = 1, 2$ *let* $R_i \subset Q_i$ *with* $\delta_i(R_i), \mu_i(R_i) \notin \mathbb{N}_0$. *Then*
(1) $R_1 \cup R_2 \in \mathfrak{C}_1^0 \cap \mathfrak{C}_2^0 \subset (\mathfrak{D}_1 \cap \mathfrak{D}_2)\backslash(\mathfrak{A}_1 \cup \mathfrak{A}_2)$.
(2) *In the closure space* $(M, (\mathfrak{A}_1 \cup \mathfrak{A}_2) \cup \{M\})$ *the set* $R_1 \cup R_2$ *is a generating set of* M *but does not contain a basis of* M.
(3) *In the closure space* $(M, (\mathfrak{D}_1 \cap \mathfrak{D}_2) \cup \{M\})$ *the set* $R_1 \cup R_2$ *is independent but cannot be completed to a basis of* M.

Proof. (2), (3) For $i \in \{1, 2\}$ and all $S \in \mathfrak{A}_i^0$, $y \in M\backslash S$, $T \in \mathfrak{C}_1^0 \cap \mathfrak{C}_2^0$ it holds $(S \cup \{y\}) \cap T = \emptyset$. □

Example 4.14. Let $M := \mathbb{R}$, $Q_1 := \mathbb{Q}$ and $Q_2 = \mathbb{R}\backslash\mathbb{Q}$. By Lemma 4.10 the set $\mathbb{Q} \cup \{\pi\}$ as well as the set $(\mathbb{R}\backslash\mathbb{Q}) \cup \{1\}$ is a basis of $(\mathbb{R}, \mathfrak{F} \cup \{\mathbb{R}\})$. Hence we have two bases with distinct cardinality and Theorem B is not valid.

References

[1] Birkhoff, G.: Lattice Theory. A.M.S. Colloq. Publ., Providence (1967)

[2] Buekenhout, F.: Espaces à fermeture. Bull. Soc. Math. Belg. **19**, 147–178 (1967)

[3] Cohn, P.M.: Universal Algebra. D. Reidel, Dordrecht (1981)

[4] Delandtsheer, A.: Dimensional linear spaces. In: Buekenhout, F. (ed.) Handbook of Incidence Geometry, chapt. 6. Elsevier, Amsterdam (1995)

[5] Karzel, H., Sörensen, K., Windelberg, D.: Einführung in die Geometrie. UTB Vandenhoeck, Göttingen (1973)

[6] Kreuzer, A., Sörensen, K.: Exchange properties in closure spaces. J. Geom. **98**, 127–138 (2010)

[7] Schmidt, J.: Einige grundlegende Begriffe und Sätze aus der Theorie der Hüllenoperatoren. Ber. Math. Tagung Berl. 21–48 (1953)

[8] Tukey, J.-W.: Convergence and Uniformity in Topology. Princeton University Press, Princeton (1940)

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