Result.Math. *Online First*

DOI 10.1007/s00025-009-0430-9

Results in Mathematics

Automorphisms of Symmetric and Double Symmetric Chain Structures

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Dedicated to Mario Marchi

Abstract. We give a description of automorphisms of symmetric and double symmetric chain structures. We use our results for double symmetric 1,2,3-structures to shed some new light on their groups of automorphisms.

Mathematics Subject Classification (2000). 51B20.

Keywords. Symmetric and double symmetric chain structures, automorphisms.

Introduction

A chain structure $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ is a net $(P, \mathfrak{G}_1, \mathfrak{G}_2)$ together with a set $\mathfrak{K}$ of chains. A subset $C \subseteq P$ is called a *chain* if C intersects each generator $X \in \mathfrak{G}_1 \cup \mathfrak{G}_2$ in exactly one point. If $\mathfrak{C}$ denotes the set of all chains of the net $(P, \mathfrak{G}_1, \mathfrak{G}_2)$ then $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{C})$ is called *maximal chain structure*. Chain structures $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ can be classified by claiming properties in particular incidence axioms and in this frame one can characterize webs, 2-structures, hyperbola structures or Minkowski planes. Another aspect of chain structures is the fact that one can associate to each pair (A, B) of chains in a natural way a permutation $\widetilde{AB}$ of P: For $p \in P$ and $i \in \{1, 2\}$ let $[p]_i$ denote the generator of $\mathfrak{G}_i$ passing through p then

$$\widetilde{AB}(p) := \big[[p]_1 \cap A\big]_2 \cap \big[[p]_2 \cap B\big]_1 .$$

For $A = B$ we set $\widetilde{A} := \widetilde{AA}$ and call $\widetilde{A}$ a *reflection in the chain* A. The map $\widetilde{AB}$ has the nice property that the image $\widetilde{AB}(C)$ of a chain is again a chain. Therefore $\tau : \mathfrak{C}^3 \to \mathfrak{C}$ with $\tau(A, B, C) := \widetilde{AC}(B)$ is a ternary operation on the set $\mathfrak{C}$ of all chains and if one fixes a chain $E \in \mathfrak{C}$ then $(\mathfrak{C}, \cdot)$ with $A \cdot B := \tau(A, E, B)$ becomes a group which is isomorphic to the symmetric group $SymE$ (cf. 1.2). These facts

allow us to define the class of *double symmetric* and of *symmetric* chain structures $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ by:

$$\forall A, B, C \in \mathfrak{K} : \tau(A, B, C) \in \mathfrak{K} \quad \text{and} \quad \forall A, B \in \mathfrak{K} : \tau(A, B, A) \in \mathfrak{K}$$

respectively which can be also characterized by:

"$(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ is double symmetric $\Leftrightarrow$ for $K \in \mathfrak{K}, K^{-1} \cdot \mathfrak{K}$ is a subgroup of $(\mathfrak{C}, \cdot)$"
(cf. 1.5) and

"$(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ is symmetric $\Leftrightarrow$ $\forall K \in \mathfrak{K} : K \cdot \mathfrak{K} \cdot K = \mathfrak{K}$ and $\mathfrak{K}^{-1} = \mathfrak{K}$ "
(cf. Theorem 2.1.(8)) .

In this paper we are interested in automorphism groups $\Gamma^+(\mathfrak{K})$, $\Gamma(\mathfrak{K})$ of a chain structure $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ which are subgroups of $\Gamma^+ := Aut(P, \Box)$ where $a \Box b := [a]_1 \cap [b]_2$ and of $\Gamma := \Gamma^+ \cup \Gamma^-$ where Γ^- consists of all antiautomorphisms of $(P, \Box)$. Hence: $\Gamma(\mathfrak{K}) := \{\gamma \in \Gamma \mid \gamma(\mathfrak{K}) = \mathfrak{K}\}$ or $\Gamma^+(\mathfrak{K}) := \{\gamma \in \Gamma^+ \mid \gamma(\mathfrak{K}) = \mathfrak{K}\}$.

If $\Gamma(\mathfrak{K})$ contains a subgroup Ξ acting transitively on $\mathfrak{K}$ then $\Gamma(\mathfrak{K})$ is determined by Ξ and the stabilizer $\Gamma_E(\mathfrak{K}) := \{\gamma \in \Gamma(\mathfrak{K}) \mid \gamma(E) = E\}$ (cf. 2.1.(8)). This is the case if the chain structure is double symmetric (cf. 3.1.(4)). Then $\Gamma(\mathfrak{K})$ is essentially determined by $\Gamma_E(\mathfrak{K}) = \Gamma_E^+(\mathfrak{K}) \circ \{id, \widetilde{E}\}$ and by 3.2.(4), $\Gamma_E^+(\mathfrak{K})$ is isomorphic to $\mathfrak{N}_{\mathfrak{C}}(\mathfrak{K})$ the normalizer of $\mathfrak{K}$ in the group $(\mathfrak{C}, \cdot)$.

Between chain structures and permutation sets there is a one-to-one correspondence via a bijection $\Pi_E : \mathfrak{C} \to SymE$ which maps each chain $C \in \mathfrak{C}$ onto a permutation of the symmetric group $SymE$ of the point set of a fixed chain $E \in \mathfrak{C}$ (cf. 1.1.(2),(3) and 1.2.(1)) and the inverse map κ_E.

In Section 3 we discuss the automorphism groups of Double Symmetric 1-,2- and 3-Structures $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$. For this purpose let $\mathbf{K} := \Pi_E(\mathfrak{K}), \mathbf{N} := \Pi_E(\mathfrak{N}_{\mathfrak{C}}(\mathfrak{K}))$.

The double symmetric 1-structures $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ are exactly the webs satisfying the Reidemeister condition and the map Π_E (cf. 1.1.(2)) takes $\mathfrak{K}$ onto a permutation group $\mathbf{K}$ acting regularly on the point set of the chain E. This allows us to turn E into a group $(E, +)$ and then $\mathfrak{N}_{\mathfrak{C}}(\mathfrak{K})$ is isomorphic to the semidirect product $(E, +) \rtimes Aut(E, +)$.

Double symmetric 2-structures $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ satisfy the rectangle axiom and correspond to near domains $(E, +, *)$. The group $(\mathfrak{K}, \cdot)$ is isomorphic to the affine group $T_2(E)$ of the near domain consisting of the maps $\tau_{m,n}(x) = m + n * x$ with $m, n \in E, n \neq o$. The permutation set $(E, \mathbf{K}) = (E, T_2(E))$ is a group acting sharply 2-transitive on the set E.

Also the 3-structures $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ - also called *hyperbola structures* - which are double symmetric are characterized by the rectangle axiom. Here the pair $(E, \mathbf{K})$ is a group acting sharply 3-transitive on the set E. Therefore after fixing an element $\infty \in E$ the set $F := E \setminus \infty$ can be furnished with an addition $+$ and a multiplication $*$ such that $(F, +, *)$ becomes a neardomain and moreover there is a certain permutation ϵ of F such that $(F, +, *, \epsilon)$ becomes even KT-field.

Finally in Section 4 we consider the larger class of symmetric chain structures $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ where we make always the assumption $E \in \mathfrak{K}$ and $(\mathfrak{C}, \cdot) := (\mathfrak{C}, \cdot\,; E)$. Then $\mathfrak{K}^{-1} = \mathfrak{K}$ and $\forall K \in \mathfrak{K} : K \cdot \mathfrak{K} \cdot K = \mathfrak{K}$ (cf. 4.1.). To the set $\mathfrak{K}$ of chains one can associate various other subsets of chains of $\mathfrak{C}$ (cf. Definition 4.2.) which are needed in order to describe the automorphism groups of the symmetric chain structure (cf. Theorem 4.4.).

Symmetric 1-Structures $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ are webs where each reflection in a chain $K \in \mathfrak{K}$ is an automorphism of the web. These were studied in several papers (cf. [2, 4]) and they correspond to the class of K-loops or what is the same to the Bruck loops. In a forthcoming paper we will investigate more thoroughly the class of symmetric 2-structures.

1. Notations and known results

1.1. Properties of maximal chain structures

First we collect some properties of maximal chain structures $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{C})$ (cf. for instance [9](2.7.5)). We recall that all chains $C \in \mathfrak{C}$ and all generators $X \in \mathfrak{G}_1 \cup \mathfrak{G}_2$ have the same cardinality which we call the *order of the chain structure* and that the cardinality of the point set P is the square of the order. To any non empty set $\mathcal{E}$ there corresponds a 2-*net* $\kappa(\mathcal{E}) = (P, \mathfrak{G}_1, \mathfrak{G}_2)$ where $P := \mathcal{E} \times \mathcal{E}$ is the product set, $\mathfrak{G}_1 := \{\{a\} \times \mathcal{E} \mid a \in \mathcal{E}\}$ and $\mathfrak{G}_2 := \{\mathcal{E} \times \{a\} \mid a \in \mathcal{E}\}$ with the binary operation $\Box : P \times P \to P; (a = (a_1, a_2), b = (b_1, b_2)) \mapsto a \Box b := (a_1, b_2)$, and the diagonal $E := \{(x, x) \mid x \in \mathcal{E}\}$.

For $a \in P$ let $[a]_1 := a \Box P$ and $[a]_2 := P \Box a$.

If $\sigma \in SymE$ is a permutation of E then the subset $\kappa(\sigma) := \{x \Box \sigma(x) \mid x \in E\}$ of P is a chain of the net. Hence the diagonal E is a chain and $E = \kappa(id)$. For $\Sigma \subseteq SymE$ let $\kappa(\Sigma) := \{\kappa(\sigma) \mid \sigma \in \Sigma\}$ and $\kappa(E, \Sigma) := (P, \mathfrak{G}_1, \mathfrak{G}_2, \kappa(\Sigma))$. $\kappa(E, \Sigma)$ is a chain structure and a maximal chain structure if $\Sigma = SymE$.

With a maximal chain structure $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{C})$ there are connected the following sets of maps:

$$\begin{aligned}
\Gamma^+ &:= Aut(P, \Box) := \big\{\sigma \in SymP \mid \forall x, y \in P : \sigma(x \Box y) = \sigma(x) \Box \sigma(y)\big\} \\
&\quad = Aut(P, \mathfrak{G}_1, \mathfrak{G}_2)\,. \\
\Gamma^- &:= \big\{\sigma \in SymP \mid \forall x, y \in P : \sigma(x \Box y) = \sigma(y) \Box \sigma(x)\big\} \\
&\quad \text{(the set of antiisomorphism)}\,. \\
\Gamma &:= \Gamma^+ \cup \Gamma^- = Aut(P, \mathfrak{G}_1 \cup \mathfrak{G}_2)\,. \\
Aut(P, \mathfrak{C}) &:= \{\sigma \in SymP \mid \forall C \in \mathfrak{C} : \sigma(C) \in \mathfrak{C}\}\,. \\
\Gamma_i &:= \Big\{\sigma \in \Gamma^+ \mid \forall x \in P : \big[\sigma(x)\big]_i = [x]_i\Big\}, \quad \text{for} \quad i = 1 \quad \text{or} \quad i = 2 \\
&\quad \text{(the elements of } \Gamma_i \text{ are called } i\text{-}maps)\,.
\end{aligned}$$

If $p \in P$ and $C \in \mathfrak{C}$, let $pC := [p]_1 \cap C$ and $Cp := [p]_2 \cap C$. For any $A, B \in \mathfrak{C}$ we have a map $\widetilde{AB} : P \to P; \widetilde{AB} : x \mapsto (Bx)\Box(xA)$. We denote $\widetilde{A} := \widetilde{AA}$. Moreover we have:

Theorem 1.1. *Let $A, B, C \in \mathfrak{C}$ then:*

1. $\widetilde{AB} \in SymP$ *with* $\widetilde{AB}(A) = B, \widetilde{AB}(B) = A$ *and* $\widetilde{AB}^{-1} = \widetilde{BA}$.
2. *The map* $\Pi_E : \mathfrak{C} \to SymE; C \mapsto \widetilde{CE} \circ \widetilde{CE}_{|E}$ *is a bijection* (*hence the cardinality of* $\mathfrak{C}$ *and of* $SymE$ *are equal*).
3. $\kappa_E : SymE \to \mathfrak{C}; \sigma \mapsto \{x\Box\sigma(x) \mid x \in E\}$ *is the inverse bijection of* Π_E (*cf.* [1, Sec.4]).
4. $\widetilde{AB}(C) \in \mathfrak{C}$, *i.e.* $\widetilde{AB}$ *induces a permutation* $\widetilde{AB}_{\mathfrak{C}}$ *of* $Sym\mathfrak{C}$.
5. *"*$\widetilde{AB}$ *is involutory* $\Leftrightarrow A = B$*"*.
6. $Fix\widetilde{AB} = A \cap B$ *in particular* $Fix\ \widetilde{A} = A$.

1.2. Notations concerning groups

Fixing an element $E \in \mathfrak{C}$ we can define by 1.1.(4) on $\mathfrak{C}$ a binary operation:

$$\cdot : \mathfrak{C} \times \mathfrak{C} \ \to \ \mathfrak{C} \ ; \ (A, B) \mapsto \widetilde{AB}(E)$$

and for $(\mathfrak{C}, \cdot\,) := (\mathfrak{C}, \cdot\,; E)$ we have:

Theorem 1.2. $(\mathfrak{C}, \cdot\,) := (\mathfrak{C}, \cdot\,; E)$ *is a group where E is the neutral element such that:*

1. Π_E *is an isomorphism from* $(\mathfrak{C}, \cdot\,)$ *onto the symmetric group* $SymE$ *and* κ_E *is the inverse isomorphism.*
2. *The map* $\widetilde{AB}_{\mathfrak{C}}$ *defined in* 1.1.(4) *has the representation* (*cf.* [9, (2.8)])*:*
$$\widetilde{AB}_{\mathfrak{C}} \ : \ \mathfrak{C} \ \to \ \mathfrak{C} \ ; \ C \ \mapsto \ A \cdot C^{-1} \cdot B \, .$$
3. $\widetilde{AB} \circ \widetilde{CD} \circ \widetilde{FG} = (\widetilde{AD^{-1}F)(GC^{-1}B})$. (*cf.* [9, (2.10.2)])
4. $\widetilde{A} \circ \widetilde{B} \circ \widetilde{A} = \widetilde{A \cdot B^{-1} \cdot A} = \widetilde{\widetilde{A}(B)}$. (*cf.* [9, (2.10.4)]).
5. $\widetilde{E}(A) = A^{-1}$.
6. *If the order of* $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{C})$ *is greater than* 2 *then the representation "*$\widetilde{AB} \mapsto \widetilde{AB}_{\mathfrak{C}}$*" is faithful. More exactly:*
   (i) *If* $\widetilde{AB}(X) = \widetilde{CD}(X)$ *for all* $X \in \mathfrak{C}$, *then* $A = C$ *and* $B = D$.
   (ii) *If* $\widetilde{AB} \circ \widetilde{E}(X) = \widetilde{CD} \circ \widetilde{E}(X)$ *for all* $X \in \mathfrak{C}$, *then* $A = C$ *and* $B = D$.
   (iii) *Any automorphism and any anti-automorphism induce different bijections of the set of chains.*

Proof. (6) (i) $\widetilde{AB}(X) = \widetilde{CD}(X) \Leftrightarrow A \cdot X^{-1} \cdot B = C \cdot X^{-1} \cdot D \Leftrightarrow C^{-1} \cdot A \cdot X^{-1} \cdot B \cdot D^{-1} = X^{-1}$, hence it is enough to prove:

$$\big(\forall X \in \mathfrak{C} : \ \widetilde{AB}(X) = X^{-1}\big) \Rightarrow A = B = E \, .$$

From $\widetilde{AB}(E) = E$ we have $B = A^{-1}$. If $\widetilde{AA^{-1}}(X) = X^{-1}$, then $A \cdot X^{-1} = X^{-1} \cdot A$, hence $A \in \mathfrak{Z}(X^{-1})$ where $\mathfrak{Z}(X^{-1})$ is a centralizer of X^{-1}. Thus $A \in \mathfrak{Z}(\mathfrak{C})$.

For $|E| > 2$ the center of $SymE$ is trivial and so the center of $(\mathfrak{C}, \cdot)$ since these groups are isomorphic by (1).

(iii)

$$\widetilde{AB} \circ \widetilde{E}(X) = \widetilde{CD}(X) \Leftrightarrow \widetilde{DC} \circ \widetilde{AB} \circ \widetilde{E}(X) = X \Leftrightarrow (D \cdot \widetilde{B^{-1})(A^{-1}} \cdot C)(X) = X\,,$$

thus it is enough to proof:

$$\forall A, B \in \mathfrak{C}: \ \exists X \in \mathfrak{C}: \ \widetilde{AB}(X) \neq X\,.$$

From $\widetilde{AB}(E) = E$ and $\widetilde{AB}(B) = B$ it follows that $B = A^{-1} = A$, thus $\widetilde{AB} = \widetilde{A}$ and A is an involution in $(\mathfrak{C}, \cdot)$. □

If we fix besides E a further chain $E' \in \mathfrak{C}$ then there are chains $A, B \in \mathfrak{C}$ such that $E' = \widetilde{AB}(E) = A \cdot B$ and with E' we define on $\mathfrak{C}$ the further binary operation $(\mathfrak{C}, \centerdot) := (\mathfrak{C}, \cdot\,; E')$ hence $X \centerdot Y := \widetilde{XY}(E')$. Then:

Theorem 1.3.

1. *$\widetilde{AB}$ induces an anti-isomorphism and $\widetilde{AB} \circ \widetilde{E}$ an isomorphism from $(\mathfrak{C}, \cdot\,, E)$ onto $(\mathfrak{C}, \cdot\,, E')$.*
2. *$\widetilde{AB}$ induces an anti-automorphism and $\widetilde{AB} \circ \widetilde{E}$ an automorphism on $(\mathfrak{C}, \cdot\,, E)$ if and only if $B = A^{-1}$. Hence if $B = A^{-1}$ then $\widetilde{AB} \circ \widetilde{E}$ is the inner automorphism $X \mapsto A \cdot X \cdot A^{-1}$.*
3. *If the order of the chain structures $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{C})$ is not 6 then each automorphism of $(\mathfrak{C}, \cdot\,, E)$ is an inner automorphism.*

Proof. (1) By 1.2.(2), $X \centerdot Y = \widetilde{XY}(E') = \widetilde{XY}(A \cdot B) = X \cdot B^{-1} \cdot A^{-1} \cdot Y$ and so $\widetilde{AB}(X \cdot Y) = A \cdot Y^{-1} \cdot X^{-1} \cdot B = (A \cdot Y^{-1} \cdot B) \centerdot (A \cdot X^{-1} \cdot B) = \widetilde{AB}(Y) \centerdot \widetilde{AB}(X)$, i.e. $\widetilde{AB}$ is an anti-isomorphism from $(\mathfrak{C}, \cdot\,, E)$ onto $(\mathfrak{C}, \centerdot, E')$. By 1.3.(1), $\widetilde{E}$ is an antiautomorphism of $(\mathfrak{C}, \cdot\,, E)$ hence $\widetilde{AB} \circ \widetilde{E}$ is an isomorphism from $(\mathfrak{C}, \cdot\,, E)$ onto $(\mathfrak{C}, \centerdot, E')$.

(3) By 1.2.(1), the group $(\mathfrak{C}, \cdot\,, E)$ is isomorphic to the symmetric group $SymE$ and any automorphism of $SymE$ is an inner automorphism if $|E| \neq 6$ (cf. e.g. [3, p. 175, Satz 5.5]). □

For a group in particular for our group $(\mathfrak{C}, \cdot\,) := (\mathfrak{C}, \cdot\,; E)$ we will use the following notations:

$$\iota : \mathfrak{C} \to \mathfrak{C}; X \mapsto X^{-1}\,.$$

For $C \in \mathfrak{C}$ let be

" $C_l : \mathfrak{C} \to \mathfrak{C}\ ;\ X \ \mapsto\ C \cdot X$ " the *left translation*,

" $C_r : \mathfrak{C} \to \mathfrak{C}\ ;\ X \ \mapsto\ X \cdot C$ " the *right translation* and

" $i_C : \mathfrak{C} \to \mathfrak{C}\ ;\ X \ \mapsto\ C \cdot X \cdot C^{-1}$ " the *inner automorphism* of $(\mathfrak{C}, \cdot\,)$

and if $\mathfrak{K} \subseteq \mathfrak{C}$ is a subset, we set $\mathfrak{K}_l := \{K_l \mid K \in \mathfrak{K}\}$, $\mathfrak{K}_r := \{K_r \mid K \in \mathfrak{K}\}$, $i_{\mathfrak{K}} := \{i_K \mid K \in \mathfrak{K}\}$, $\widetilde{\mathfrak{K}} := \{\widetilde{K} \mid K \in \mathfrak{K}\}$ and denote by $\mathfrak{N}_{\mathfrak{C}}(\mathfrak{K}) := \{C \in \mathfrak{C} \mid C \cdot \mathfrak{K} \cdot C^{-1} = \mathfrak{K}\}$ *the normalizer* of $\mathfrak{K}$ in $\mathfrak{C}$.

If $\mathfrak{A}, \mathfrak{B} \leq Sym\mathfrak{C}$ are two subgroups then we denote by $\mathfrak{A} \rtimes \mathfrak{B}$ the semidirect product, i.e. $\mathfrak{A} \cap \mathfrak{B} = \{id\}$ and $\mathfrak{B} \subseteq \mathfrak{N}_{\mathfrak{C}}(\mathfrak{A})$ and by $\mathfrak{A} \oplus \mathfrak{B}$ the direct product, i.e. $\mathfrak{A} \rtimes \mathfrak{B}$ and $\mathfrak{B} \rtimes \mathfrak{A}$. We denote $\widehat{\mathfrak{C}} := \langle \mathfrak{C}_l, \mathfrak{C}_r, \iota \rangle = (\mathfrak{C}_l \circ \mathfrak{C}_r) \rtimes \{id, \iota\}$ the subgroup of $Sym\mathfrak{C}$.

For $A, B \in \mathfrak{C}$ the map $\widetilde{AB}_{\mathfrak{C}}$ can be written in the form:

$$\widetilde{AB}_{\mathfrak{C}} = (A \cdot B)_r \circ i_A \circ \iota = (A \cdot B)_l \circ i_{B^{-1}} \circ \iota\,.$$

Now we can state for a maximal chain structures $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{C})$ the following theorem (cf. [9, (2.4),(2.11)]) where $(\mathfrak{C}, \cdot\,) := (\mathfrak{C}, \cdot\,; E)$:

Theorem 1.4. *If $|E| > 2$ then the map*

$$\psi : \Gamma \to \widehat{\mathfrak{C}};\ \widetilde{AB} \circ \widetilde{E} \mapsto A_l \circ B_r,\ \widetilde{AB} \mapsto A_l \circ B_r \circ \iota$$

is an isomorphism and:

1. *For $A, B, C \in \mathfrak{C}$ we have: $C_l = \psi(\widetilde{CE} \circ \widetilde{E})$, $C_r = \psi(\widetilde{EC} \circ \widetilde{E})$, $A \cdot B = A_l(B) = B_r(A)$, $(A \cdot B)_l = A_l \circ B_l$, $(A \cdot B)_r = B_r \circ A_r$, $A_l \circ B_r = B_r \circ A_l = \psi(\widetilde{AB} \circ \widetilde{E})$, and $\Pi_E(C) = \psi^{-1}(C_l \circ C_r^{-1})_{|E}$.*
2. *$(\mathfrak{C}_l, \circ)$ and $(\mathfrak{C}_r, \circ)$ are subgroups of $Sym\mathfrak{C}$ where $(\mathfrak{C}_l, \circ)$ is isomorphic and $(\mathfrak{C}_r, \circ)$ antiisomorphic to $(\mathfrak{C}, \cdot\,)$ and $\mathfrak{C}_l \circ \mathfrak{C}_r = \mathfrak{C}_l \oplus \mathfrak{C}_r$ is the direct product.*
3. *$\Gamma = Aut(P, \mathfrak{C}) = \Gamma^- \,\dot\cup\, \Gamma^+$ and $\iota = \psi(\widetilde{E})$.*
4. *$\Gamma^- = \{\widetilde{AB} \mid A, B \in \mathfrak{C}\}$, $\psi(\Gamma^-) = \mathfrak{C}_r \circ i_{\mathfrak{C}} \circ \iota = \mathfrak{C}_l \circ i_{\mathfrak{C}} \circ \iota$, $\Gamma^+ = \Gamma^- \circ \widetilde{E}$, $\psi(\Gamma^+) = \mathfrak{C}_l \rtimes i_{\mathfrak{C}}$.*
5. *$\Gamma_1 = \{\widetilde{AE} \circ \widetilde{E} \mid A \in \mathfrak{C}\}$, $\Gamma_2 = \{\widetilde{EA} \circ \widetilde{E} \mid A \in \mathfrak{C}\}$, $\psi(\Gamma_1) = \mathfrak{C}_l$, $\psi(\Gamma_2) = \mathfrak{C}_r$ ([9, (2.12.3)]) and $\Gamma^+ = Aut(P, \Box) = \Gamma_1 \oplus \Gamma_2$ (cf. [9, (1.2.5)]).*
6. *$\Gamma_E^- := \{\gamma \in \Gamma^- \mid \gamma(E) = E\} = \{\widetilde{CC^{-1}} \mid C \in \mathfrak{C}\}$, $\psi(\Gamma_E^-) = i_{\mathfrak{C}} \circ \iota$,*
   $\Gamma_E^+ := \{\gamma \in \Gamma^+ \mid \gamma(E) = E\} = \{\widetilde{CC^{-1}} \circ \widetilde{E} \mid C \in \mathfrak{C}\}$, $\psi(\Gamma_E^+) = i_{\mathfrak{C}}$.

1.3. Substructures

If $\emptyset \neq \mathfrak{K} \subseteq \mathfrak{C}$ then $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ is called a *substructure* of $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{C})$. Moreover following [1] sec. 6, we call $\mathfrak{K}$ *symmetric* if "$\forall A, B \in \mathfrak{K} : \widetilde{A}(B) \in \mathfrak{K}$" and *double symmetric* if "$\forall A, B, C \in \mathfrak{K} : \widetilde{AB}(C) \in \mathfrak{K}$" (cf. [9, (1.4)]). Clearly any double symmetric chain structure is symmetric.

For $E \in \mathfrak{C}$ the permutation set $\Pi_E(\mathfrak{K})$ of $SymE$ shall be called the *germ* of the chain structure $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ in E. On the other hand if $\Sigma \subseteq SymE$ then $(P, \mathfrak{G}_1, \mathfrak{G}_2, \kappa_E(\Sigma))$ is a chain structure – called *chain derivation*. If $E \in \mathfrak{K}$ we call the subgroup $\langle \Pi_E(\mathfrak{K}) \rangle$ of $SymE$ generated by the germ $\Pi_E(\mathfrak{K})$ the *von STAUDT group* of the chain structure $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ (cf. [7, p. 60]). If $F \in \mathfrak{K}$ is an other chain then the groups $\langle \Pi_E(\mathfrak{K}) \rangle$ and $\langle \Pi_F(\mathfrak{K}) \rangle$ are isomorphic but not necessarly the germs $\Pi_E(\mathfrak{K})$ and $\Pi_F(\mathfrak{K})$. We note:

Theorem 1.5. *Let $\emptyset \neq \mathfrak{K} \subseteq \mathfrak{C}$ then:*

1. *$\mathfrak{K}$ is double symmetric $\Leftrightarrow$ For $K \in \mathfrak{K}$: $\mathfrak{K} \cdot K^{-1} \leq (\mathfrak{C}, \cdot\,)$, i.e. $\mathfrak{K}$ is the coset of a subgroup of $(\mathfrak{C}, \cdot\,)$.*
2. *If $E \in \mathfrak{K}$ then: $\mathfrak{K}$ is double symmetric $\Leftrightarrow \mathfrak{K} \leq (\mathfrak{C}, \cdot\,)$.*

For $A, B \in \mathfrak{K}$ and $i \in \{1, 2\}$ we consider the i-perspectivities

$$[A \xrightarrow{i} B] : A \to B; a \mapsto [a]_i \cap B \quad \text{(cf. [9, (1.4.3)])}$$

and call a map π which can be decomposed into a product of i-perspectivities a *projectivity* of $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$. For $E \in \mathfrak{K}$ the set of all projectivities $\pi : E \to E$ mapping E onto E forms a group which coincides with the von STAUDT group $\langle \Pi_E(\mathfrak{K}) \rangle$ of the chain structure $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ (cf. [7, p. 60]). Furthermore we have:

Theorem 1.6. *$\kappa_A(\langle \Pi_A(\mathfrak{K}) \rangle) = \kappa_B(\langle \Pi_B(\mathfrak{K}) \rangle)$, i.e. the chain derivations of the von Staudt groups belonging to the chains A and B result in the same chain structure $(P, \mathfrak{G}_1, \mathfrak{G}_2, \overline{\mathfrak{K}})$ with $\overline{\mathfrak{K}} = \kappa_A(\langle \Pi_A(\mathfrak{K}) \rangle) = \kappa_B(\langle \Pi_B(\mathfrak{K}) \rangle)$ having the following properties:*

(1) *$\forall K, L, M \in \overline{\mathfrak{K}} : \widetilde{KL}(M) \in \overline{\mathfrak{K}}$. (I.e. $(P, \mathfrak{G}_1, \mathfrak{G}_2, \overline{\mathfrak{K}})$ is the smallest double symmetric chain structure with $\mathfrak{K} \subseteq \overline{\mathfrak{K}}$).*
(2) *If $E \in \overline{\mathfrak{K}}$ and $(\mathfrak{C}, \cdot) := (\mathfrak{C}, \cdot; E)$ then $\overline{\mathfrak{K}}$ is a subgroup of $(\mathfrak{C}, \cdot)$ and $\overline{\mathfrak{K}}$ is generated by $\mathfrak{K}$.*

Definition 1.7. The chain structure $(P, \mathfrak{G}_1, \mathfrak{G}_2, \overline{\mathfrak{K}})$ formed according to 1.6. is called the *group envelope* or the *double symmetric envelope* of the chain structure $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$.

Theorem 1.8. *Let $\mathfrak{K} \subseteq \mathfrak{C}$ then:*

(1) *$(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{N}_{\mathfrak{C}}(\mathfrak{K}))$ is a double symmetric chain structure,*
(2) *If $\mathfrak{K}$ is double symmetric and $E \in \mathfrak{K}$ then $\mathfrak{K} \trianglelefteq \mathfrak{N}_{\mathfrak{C}}(\mathfrak{K})$ hence $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ is a substructure of $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{N}_{\mathfrak{C}}(\mathfrak{K}))$ and $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{N}_{\mathfrak{C}}(\mathfrak{K}))$ is called the normal envelope .*

1.4. Sharply n transitive permutation sets and their chain structures

We recall, if $(P, \mathfrak{G}_1, \mathfrak{G}_2)$ is a 2-net then a subset $A \subseteq P$ is called *joinable* if $\forall X \in \mathfrak{G}_1 \cup \mathfrak{G}_2 : |X \cap A| \leq 1$ and for $n \in \mathbf{N}$ we set $P^{(n)} := \{(a_1, a_2, \ldots, a_n) \in P^n \mid \{a_1, a_2, \ldots, a_n\}$ is joinable set of pairwise different points$\}$.

Definition 1.9. A chain structure $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ is called a *n-structure* if the axiom:

(n). $\forall (p_1, p_2, \ldots, p_n) \in P^{(n)} \exists_1 K \in \mathfrak{K} : p_1, p_2, \ldots, p_n \in K$.

is satisfied. Then for $n = 1$, $n = 2$ and $n = 3$ $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ is called a *web*, a 2-*struture* and a *hyperbola structure* respectively.

A *Minkowski planes* is a hyperbola structure $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{M})$ satisfying the *touching axiom*:

(T) $\forall M \in \mathfrak{M}$, $\forall m \in M$, $\forall p \in P \setminus (M \cup [m]_1 \cup [m]_2) : \exists_1 N \in \mathfrak{M} : p \in N$ and $N \cap M = \{m\}$.

Theorem 1.10. *Let $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ be a chain structure, let $E \in \mathfrak{K}$ be fixed and let $\Sigma := \Pi_E(\mathfrak{K})$ be its germ in E then:*

1. *(E, Σ) is a sharply n transitive permutation set $\Leftrightarrow$ $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ is a n-structure.*

2. (E, Σ) *is a permutation group* $\Leftrightarrow (P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ *is double symmetric.*
3. (E, Σ) *is a sharply n transitive permutation group* $\Leftrightarrow (P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ *is a double symmetric n-structure.*

1.5. The symmetric stabilizer of a chain structure

Definition 1.11. For any $\mathfrak{K} \subseteq \mathfrak{C}$ the set $\mathfrak{K}^s := \{C \in \mathfrak{C} \mid \widetilde{C}(\mathfrak{K}) = \mathfrak{K}\}$ is called *symmetric stabilizer of* $\mathfrak{K}$ and the chain structure $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K}^s)$ is called the *symmetric stabilizer of* $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$.

Theorem 1.12. *For* $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K}^s)$ *we have:*

1. $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K}^s)$ *is a symmetric chain structure.*
2. *If* $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ *is symmetric then* $\mathfrak{K} \subseteq \mathfrak{K}^s$.
3. $\mathfrak{K}^s \subseteq (\mathfrak{K}^s)^s$.
4. *If* $E \in \mathfrak{K}$ *and* $(\mathfrak{C}, \cdot\,; E)$ *is turned in a group then* $\mathfrak{K}^s \subseteq \{A \in \mathfrak{C} \mid A^2 \in \mathfrak{K}\}$.

Proof. (1) Let $A, B \in \mathfrak{K}^s$ then $\widetilde{\widetilde{A}(B)} = \widetilde{A} \circ \widetilde{B} \circ \widetilde{A}$ and so $\widetilde{\widetilde{A}(B)}(\mathfrak{K}) = \widetilde{A} \circ \widetilde{B} \circ \widetilde{A}(\mathfrak{K}) = \widetilde{A} \circ \widetilde{B}(\mathfrak{K}) = \widetilde{A}(\mathfrak{K}) = \mathfrak{K}$, i.e. $\widetilde{A}(B) \in \mathfrak{K}^s$. (3) By (1) and (2), $\mathfrak{K}^s \subseteq (\mathfrak{K}^s)^s$. (4) follows from $\widetilde{A}(E) \in \mathfrak{K}$. □

2. Automorphisms of a chain structure $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$

In this section we will consider for an arbitrary chain structure $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ the following automorphism groups:

$$\begin{aligned}
\Gamma(\mathfrak{K}) &:= Aut(P, \mathfrak{G}_1 \cup \mathfrak{G}_2, \mathfrak{K}) = \left\{\gamma \in \Gamma \mid \gamma(\mathfrak{K}) = \mathfrak{K}\right\}, \\
\Gamma^+(\mathfrak{K}) &:= Aut(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K}) = \left\{\gamma \in \Gamma^+ \mid \gamma(\mathfrak{K}) = \mathfrak{K}\right\}, \\
\Gamma^-(\mathfrak{K}) &:= Aut(P, \mathfrak{G}_1 \cup \mathfrak{G}_2, \mathfrak{K})^- = \left\{\gamma \in \Gamma^- \mid \gamma(\mathfrak{K}) = \mathfrak{K}\right\}, \\
\Gamma_i(\mathfrak{K}) &:= Aut(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})_i = \left\{\gamma \in \Gamma_i \mid \gamma(\mathfrak{K}) = \mathfrak{K}\right\} \quad \text{for} \quad i = 1, 2\,. \\
\Gamma_E^+(\mathfrak{K}) &:= \left\{\gamma \in \Gamma^+(\mathfrak{K}) \mid \gamma(E) = E\right\}, \\
\Gamma_E^-(\mathfrak{K}) &:= \left\{\gamma \in \Gamma^-(\mathfrak{K}) \mid \gamma(E) = E\right\}, \\
\Gamma_E(\mathfrak{K}) &:= \left\{\gamma \in \Gamma(\mathfrak{K}) \mid \gamma(E) = E\right\} = \Gamma_E^+(\mathfrak{K}) \dot\cup \Gamma_E^-(\mathfrak{K})\,.
\end{aligned}$$

For this purpose let $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{C})$ be the corresponding maximal chain structure, let $E \in \mathfrak{K}$ be fixed and let $(\mathfrak{C}, \cdot) := (\mathfrak{C}, \cdot, E)$ be the group defined according to 1.2.(2) by $A \cdot B := \widetilde{AB}(E)$. To characterize these automorphism groups we need the additional notation: ${}^{\bullet}\mathfrak{K} := \{C \in \mathfrak{C} \mid C \cdot \mathfrak{K} = \mathfrak{K}\}$, $\mathfrak{K}^{\bullet} := \{C \in \mathfrak{C} \mid \mathfrak{K} \cdot C = \mathfrak{K}\}$.

Theorem 2.1. $\Gamma(\mathfrak{K}) \leq \Gamma$, $\Gamma^+(\mathfrak{K}) \leq \Gamma^+, \Gamma^-(\mathfrak{K}) \subseteq \Gamma^-$, $\Gamma_i(\mathfrak{K}) \leq \Gamma_i$ *for* $i = 1, 2$. *Moreover:*

1. $\Gamma^-(\mathfrak{K}) = \{\widetilde{AB} \mid A, B \in \mathfrak{C} : A \cdot \mathfrak{K}^{-1} \cdot B = \mathfrak{K}\}$
2. $\Gamma^+(\mathfrak{K}) = \{\widetilde{AB} \circ \widetilde{E} \mid A, B \in \mathfrak{C} : A \cdot \mathfrak{K} \cdot B = \mathfrak{K}\} \supseteq \Gamma_1(\mathfrak{K}) \oplus \Gamma_2(\mathfrak{K})$,
3. $\Gamma_1(\mathfrak{K}) = \{\widetilde{AE} \circ \widetilde{E} \mid A \in {}^{\bullet}\mathfrak{K}\} \trianglelefteq \Gamma^+(\mathfrak{K})$,
4. $\Gamma_2(\mathfrak{K}) = \{\widetilde{EA} \circ \widetilde{E} \mid A \in \mathfrak{K}^{\bullet}\} \trianglelefteq \Gamma^+(\mathfrak{K})$,

5. *If* $\widetilde{E}(\mathfrak{K}) = \mathfrak{K}$ *then* $\mathfrak{K}^{-1} = \mathfrak{K}$, $\Gamma^-(\mathfrak{K}) = \{\widetilde{AB} \mid A \cdot \mathfrak{K} \cdot B = \mathfrak{K}\} \supseteq (\Gamma_1(\mathfrak{K}) \oplus \Gamma_2(\mathfrak{K})) \circ \widetilde{E}$,
6. $\Gamma_E^+(\mathfrak{K}) = \{\widetilde{AA^{-1}} \circ \widetilde{E} \mid A \in \mathfrak{N}_{\mathfrak{C}}(\mathfrak{K})\}$,
7. $\Gamma_E^-(\mathfrak{K}) = \{\widetilde{AA^{-1}} \mid A \in \mathfrak{N}_{\mathfrak{C}}(\mathfrak{K})\}$,
8. $\Gamma_1(\mathfrak{K}) \rtimes \Gamma_E^+(\mathfrak{K}) \leq \Gamma^+(\mathfrak{K})$.
9. *If there is a subgroup* $\Xi \leq \Gamma(\mathfrak{K})$ *acting transitively on* $\mathfrak{K}$ *then* $\Gamma(\mathfrak{K}) = \Xi \circ \Gamma_E(\mathfrak{K})$.
10. $\mathfrak{K}$ *is symmetric* $\Leftrightarrow$ $\widetilde{\mathfrak{K}} := \{\widetilde{K} \mid K \in \mathfrak{K}\} \subseteq \Gamma^-(\mathfrak{K})$ $\Leftrightarrow$ $\forall K \in \mathfrak{K} : K \cdot \mathfrak{K} \cdot K = \mathfrak{K}$ *and* $\mathfrak{K}^{-1} = \mathfrak{K}$ $\Leftrightarrow$ $\forall \widetilde{A}, \widetilde{B} \in \widetilde{\mathfrak{K}} : \widetilde{A} \circ \widetilde{B} \circ \widetilde{A} \in \widetilde{\mathfrak{K}}$.
11. $\mathfrak{K}$ *is double symmetric* $\Leftrightarrow$ $\widetilde{\widetilde{\mathfrak{K}}} := \{\widetilde{AB} | A, B \in \mathfrak{K}\} \subseteq \Gamma^-(\mathfrak{K})$ $\Leftrightarrow$ $\mathfrak{K} \leq (\mathfrak{C}, \cdot)$.

Proof. (1), (2), and the first parts of (3) and (4) follow directly from 1.2.(2), 1.4.(4) and 1.4.(5).

The second part of (3) Let $\gamma := \widetilde{AB} \circ \widetilde{E} \in \Gamma^+(\mathfrak{K})$ (hence $A \cdot \mathfrak{K} \cdot B = \mathfrak{K}$) and $\kappa := \widetilde{CE} \circ \widetilde{E} \in \Gamma_1(\mathfrak{K})$ (hence $C \cdot \mathfrak{K} = \mathfrak{K}$) then $\gamma^{-1} \circ \kappa \circ \gamma = \widetilde{E} \circ \widetilde{BA} \circ \widetilde{CE} \circ \widetilde{E} \circ \widetilde{AB} \circ \widetilde{E} = (\widetilde{A^{-1}CA})E \circ \widetilde{E}$. Hence $\gamma^{-1} \circ \kappa \circ \gamma \in \Gamma_1(\mathfrak{K}) \Longleftrightarrow A^{-1}CA \cdot \mathfrak{K} = \mathfrak{K}$.

We have $A^{-1}CA \cdot \mathfrak{K} = A^{-1}C \cdot \mathfrak{K} \cdot B^{-1} = A^{-1} \cdot \mathfrak{K} \cdot B^{-1} = \mathfrak{K}$.

(5) Let $K \in \mathfrak{K}$ then by 1.2.(2), $\widetilde{E}(K) = E \cdot K^{-1} \cdot E = K^{-1}$ hence $\mathfrak{K}^{-1} = \mathfrak{K}$ and $\widetilde{AB}(K^{-1}) = A \cdot K \cdot B$. Thus (5) follows from (1).

(6), (7) follow directly from 1.4. (6). (8) If $\gamma := \widetilde{AB} \circ \widetilde{E} \in \Gamma^+(\mathfrak{K})$ (hence $A \cdot \mathfrak{K} \cdot B = \mathfrak{K}$) then $\widetilde{AB} \circ \widetilde{E} = ((\widetilde{AB})E \circ \widetilde{E}) \circ (\widetilde{B^{-1}B} \circ \widetilde{E}))$.

(10) If $\mathfrak{K}$ is symmetric hence "$\forall A, B \in \mathfrak{K} : \widetilde{A}(B) = A \cdot B^{-1} \cdot A \in \mathfrak{K}$" then by $E \in \mathfrak{K}$, $\widetilde{E}(\mathfrak{K}) = \mathfrak{K}^{-1} = \mathfrak{K}$ and so (10) is a consequence of (1) and (5).

(11) Since by 1.2.(2), $\widetilde{AB}(C) = A \cdot C^{-1} \cdot B$ and since $E \in \mathfrak{K}$ we have:

$$\mathfrak{K} \text{ is double symmetric} \Leftrightarrow \forall A, B, C \in \mathfrak{K} : A \cdot B^{-1} \cdot C \in \mathfrak{K} \Rightarrow$$
$$(\text{for } C = E)\ \forall A, B \in \mathfrak{K} : A \cdot B^{-1} \in \mathfrak{K} \Leftrightarrow \mathfrak{K} \leq (\mathfrak{C}, \cdot, E) \Rightarrow$$
$$\forall A, B, C \in \mathfrak{K} : \widetilde{AB}(C) = A \cdot C^{-1} \cdot B \in \mathfrak{K}. \qquad \square$$

Corollary 2.2. *Let* ψ *be the isomorphism defined in Theorem* 1.4*, then:*

1. $\psi(\Gamma_1(\mathfrak{K})) = {}^{\bullet}\mathfrak{K}_l$, $\psi(\Gamma_2(\mathfrak{K})) = \mathfrak{K}^{\bullet}{}_r$,
2. $\psi(\Gamma^+(\mathfrak{K})) \supseteq (\mathfrak{K}_r^{\bullet} \oplus {}^{\bullet}\mathfrak{K}_l)$,
3. *If* $\widetilde{E}(\mathfrak{K}) = \mathfrak{K}$ *then* $\psi(\Gamma^-(\mathfrak{K})) \supseteq (\mathfrak{K}_r^{\bullet} \oplus {}^{\bullet}\mathfrak{K}_l) \circ \iota$.

3. Automorphisms of double symmetric chain structures

In this section let $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ be a double symmetric chain structure and $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{C})$ the corresponding maximal chain structure, let $E \in \mathfrak{K}$ be fixed and let $(\mathfrak{C}, \cdot) := (\mathfrak{C}, \cdot, E)$ be the group defined according to 1.2.

Theorem 3.1. *Let $\mathfrak{K}$ be double symmetric and $E \in \mathfrak{K}$, then:*

1. $\Gamma_1(\mathfrak{K}) = \{\widetilde{KE} \circ \widetilde{E} |\ K \in \mathfrak{K}\}$ *and* $\Gamma_2(\mathfrak{K}) = \{\widetilde{EK} \circ \widetilde{E} |\ K \in \mathfrak{K}\}$ *are subgroups of* $\Gamma(\mathfrak{K})$ *each acting transitively on* $\mathfrak{K}$,
2. $\Gamma^-(\mathfrak{K}) = \{\widetilde{A(A^{-1} \cdot K)} |\ A \in \mathfrak{N}_{\mathfrak{C}}(\mathfrak{K}), K \in \mathfrak{K}\} = \Gamma_2(\mathfrak{K}) \circ \Gamma_E^-(\mathfrak{K}) = \Gamma_E^-(\mathfrak{K}) \circ \Gamma_1(\mathfrak{K})$,
3. $\Gamma^+(\mathfrak{K}) = \{\widetilde{EK} \circ \widetilde{AA^{-1}} |\ A \in \mathfrak{N}_{\mathfrak{C}}(\mathfrak{K}), K \in \mathfrak{K}\} = \Gamma_2(\mathfrak{K}) \rtimes \Gamma_E^+(\mathfrak{K}) = \Gamma_1(\mathfrak{K}) \rtimes \Gamma_E^+(\mathfrak{K})$,
4. $\Gamma(\mathfrak{K}) = \langle \widetilde{\widetilde{\mathfrak{K}}} \cup \Gamma_E^+(\mathfrak{K}) \rangle = \Gamma_2(\mathfrak{K}) \rtimes \Gamma_E(\mathfrak{K}) = \Gamma_1(\mathfrak{K}) \rtimes \Gamma_E(\mathfrak{K})$.

Proof. (1) follows from 2.1.(3) and 2.1.(4). (2) Let $A, B \in \mathfrak{C}$ with $A \cdot \mathfrak{K} \cdot B = \mathfrak{K}$ (cf. Theorem 2.1.(1), (2)). Then $A \cdot E \cdot B = A \cdot B =: K_o \in \mathfrak{K}$ hence $B = A^{-1} \cdot K_o$ and $A \cdot \mathfrak{K} \cdot A^{-1} \cdot K_o = \mathfrak{K}$. This shows $\widetilde{AB} \in \Gamma(\mathfrak{K}) \Leftrightarrow A$ is the normalizer of $\mathfrak{K}$ and $A \cdot B \in \mathfrak{K}$. Thus $\Gamma(\mathfrak{K})^- = \{\widetilde{A(A^{-1} \cdot K)} |\ A \in \mathfrak{C} : A \cdot \mathfrak{K} \cdot A^{-1} = \mathfrak{K}, K \in \mathfrak{K}\}$. Additionally we have $\widetilde{A(A^{-1} \cdot K)} = (\widetilde{EK} \circ \widetilde{E}) \circ \widetilde{AA^{-1}}$ by 1.2.(2). Therefore the rest of (2) follows from (1) and 2.1.(7). By 1.2.(5) and our assumptions, $\widetilde{E}(\mathfrak{K}) = \mathfrak{K}^{-1} = \mathfrak{K}$ hence by 1.4.(4), $\Gamma^+(\mathfrak{K}) = \Gamma^-(\mathfrak{K}) \circ \widetilde{E}$ and since (by 1.2.(3)), $\widetilde{A(A^{-1}K)} \circ \widetilde{E} = \widetilde{EK} \circ \widetilde{AA^{-1}}$, (3) follows from (2). □

Corollary 3.2. *Let $\mathfrak{K}$ be double symmetric, $E \in \mathfrak{K}$ and let ψ be the isomorphism defined in Theorem* 1.4, *then* ${}^{\bullet}\mathfrak{K} = \mathfrak{K}^{\bullet} = \mathfrak{K}$ *and:*

1. $\psi(\Gamma_1(\mathfrak{K})) = \mathfrak{K}_l$, $\psi(\Gamma_2(\mathfrak{K})) = \mathfrak{K}_r$,
2. $\psi(\langle \widetilde{\widetilde{\mathfrak{K}}} \rangle) = \mathfrak{K}_l \circ \mathfrak{K}_r \circ \{id, \iota\}$,
3. $i_{\mathfrak{K}} \trianglelefteq i_{\mathfrak{N}_{\mathfrak{C}}(\mathfrak{K})} = \psi(\Gamma_E^+(\mathfrak{K}))$,
4. $\psi(\Gamma_E(\mathfrak{K})) = i_{\mathfrak{N}_{\mathfrak{C}}(\mathfrak{K})} \oplus \{id, \iota\}$,
5. $\psi(\Gamma^-(\mathfrak{K})) = \mathfrak{K}_r \circ i_{\mathfrak{N}_{\mathfrak{C}}(\mathfrak{K})} \circ \iota$,
6. $\psi(\Gamma^+(\mathfrak{K})) = \mathfrak{K}_r \rtimes i_{\mathfrak{N}_{\mathfrak{C}}(\mathfrak{K})} = \mathfrak{K}_l \rtimes i_{\mathfrak{N}_{\mathfrak{C}}(\mathfrak{K})}$
7. $\psi(\Gamma(\mathfrak{K})) = \mathfrak{K}_r \rtimes (i_{\mathfrak{N}_{\mathfrak{C}}(\mathfrak{K})} \oplus \{\iota, id\}) = \mathfrak{K}_l \rtimes (i_{\mathfrak{N}_{\mathfrak{C}}(\mathfrak{K})} \oplus \{\iota, id\})$.

Remark 3.3. We have: $\psi(\Gamma^+(\mathfrak{K})) = \mathfrak{K}_r \rtimes i_{\mathfrak{C}} \Leftrightarrow \mathfrak{K} \trianglelefteq \mathfrak{C}$ and if $5 \leq |\mathfrak{G}_1| < \infty$ and $\mathfrak{K} \trianglelefteq \mathfrak{C}$ then $(\mathfrak{K}, \cdot)$ is isomorphic to the alternative group $Alt\mathfrak{G}_1$.

Remark 3.4. $\Gamma^+(\mathfrak{K})$ acts transitively on the set $\mathfrak{K}$ and $\Gamma_1(\mathfrak{K})$ is a normal subgroup acting regularly on the set $\mathfrak{K}$.

Corollary 3.5. *Let $\mathfrak{D}$ be a subgroup of $\mathfrak{C}$ with $\mathfrak{K} \cap \mathfrak{D} = \{E\}$. Then*

$$\mathfrak{N}_{\mathfrak{C}}(\mathfrak{K}) = \mathfrak{K} \rtimes \mathfrak{D} \Leftrightarrow \Gamma(\mathfrak{K}) = \langle \widetilde{\widetilde{\mathfrak{K}}} \rangle \rtimes \{\widetilde{DD^{-1}} \circ \widetilde{E} |\ D \in \mathfrak{D}\}.$$

In this case $\psi(\Gamma^+(\mathfrak{K})) = (\mathfrak{K}_r \rtimes i_{\mathfrak{K}}) \rtimes i_{\mathfrak{D}}$ *and* $\psi(\Gamma(\mathfrak{K})) = (\mathfrak{K}_r \rtimes (i_{\mathfrak{K}} \oplus \{\iota, id\})) \rtimes i_{\mathfrak{D}}$.

Proof. " $\Rightarrow$ " Let $C = K \cdot D$ with $K \in \mathfrak{K}$ and $D \in \mathfrak{D}$. By 1.2.(3) we have $\widetilde{CC^{-1}} = \widetilde{KK^{-1}} \circ \widetilde{E} \circ \widetilde{DD^{-1}} = \widetilde{KK^{-1}} \circ \widetilde{DD^{-1}} \circ \widetilde{E}$ and so $\Gamma_E(\mathfrak{K}) = \{\widetilde{KK^{-1}} \circ \widetilde{E} \mid K \in \mathfrak{K}\} \oplus \{\widetilde{DD^{-1}} \circ \widetilde{E} \mid D \in \mathfrak{D}\} \oplus \{id, \widetilde{E}\}$ implying $\Gamma(\mathfrak{K}) = \Gamma_1(\mathfrak{K}) \rtimes \Gamma_E(\mathfrak{K}) = \{\widetilde{KE} \circ \widetilde{E} |\ K \in \mathfrak{K}\} \rtimes (\{\widetilde{KK^{-1}} \circ \widetilde{E} \mid K \in \mathfrak{K}\} \oplus \{\widetilde{DD^{-1}} \circ \widetilde{E} \mid D \in \mathfrak{D}\} \oplus \{id, \widetilde{E}\}) =$

$(\{\widetilde{KE} \circ \widetilde{E} \mid K \in \mathfrak{K}\} \rtimes (\{\widetilde{KK^{-1}} \circ \widetilde{E} \mid K \in \mathfrak{K}\} \oplus \{id, \widetilde{E}\}) \rtimes \{\widetilde{DD^{-1}} \circ \widetilde{E} \mid D \in \mathfrak{D}\}) = \langle \widetilde{\widetilde{\mathfrak{K}}} \rangle \rtimes \{\widetilde{DD^{-1}} \circ \widetilde{E} \mid D \in \mathfrak{D}\}$.

" $\Leftarrow$ " If $M \in \mathfrak{N}_{\mathfrak{C}}(\mathfrak{K})$ then by 2.1.(7), 3.1. and assumption, $\widetilde{MM^{-1}} \in \Gamma_E(\mathfrak{K}) \leq \Gamma(\mathfrak{K}) = \langle \widetilde{\widetilde{\mathfrak{K}}} \rangle \rtimes \{\widetilde{DD^{-1}} \circ \widetilde{E} \mid D \in \mathfrak{D}\}$, i.e. $\exists_1 (A, B, D) \in \mathfrak{K} \times \mathfrak{K} \times \mathfrak{D}$ such that $\widetilde{MM^{-1}} = \widetilde{AB} \circ \widetilde{DD^{-1}} \circ \widetilde{E} = \widetilde{(AD)(D^{-1}B)}$ by 1.2.(3) hence $M = AD = B^{-1}D$ and $B = A^{-1}$, and so $\mathfrak{N}_{\mathfrak{C}}(\mathfrak{K}) \leq \mathfrak{K} \cdot \mathfrak{D}$. Now let $K \in \mathfrak{K}$, $D \in \mathfrak{D}$ and let $N := K \cdot D$ then $\widetilde{KK^{-1}} \circ \widetilde{DD^{-1}} \circ \widetilde{E} \in \Gamma(\mathfrak{K})$ and $\widetilde{KK^{-1}} \circ \widetilde{DD^{-1}} \circ \widetilde{E}(E) = E$ hence $\widetilde{KK^{-1}} \circ \widetilde{DD^{-1}} \circ \widetilde{E} = \widetilde{NN^{-1}} \in \Gamma_E^-(\mathfrak{K})$ and so by 2.1.(7), $N := K \cdot D \in \mathfrak{N}_{\mathfrak{C}}(\mathfrak{K})$, i.e. $\mathfrak{N}_{\mathfrak{C}}(\mathfrak{K}) = \mathfrak{K} \cdot \mathfrak{D}$. Finally $\widetilde{DD^{-1}} \circ \widetilde{KK^{-1}} \circ (\widetilde{DD^{-1}})^{-1} = \widetilde{(DKD^{-1})(DK^{-1}D^{-1})} \in \langle \widetilde{\widetilde{\mathfrak{K}}} \rangle$ showing $DKD^{-1} \in \mathfrak{K}$, i.e. $\mathfrak{N}_{\mathfrak{C}}(\mathfrak{K}) = \mathfrak{K} \rtimes \mathfrak{D}$. $\square$

Examples. Automorphisms of double symmetric 1-,2- and 3-structures

In the following let $E \in \mathfrak{C}$ and $(\mathfrak{C}, \cdot) := (\mathfrak{C}, \cdot, E)$, $\mathfrak{K} \leq (\mathfrak{C}, \cdot)$ and let $\mathbf{K} := \Pi_E(\mathfrak{K}), \mathbf{N} := \Pi_E(\mathfrak{N}_{\mathfrak{C}}(\mathfrak{K}))$. Then by 1.1., $\mathbf{K} \trianglelefteq \mathbf{N} \leq SymE$ and $\mathbf{N}$ is the normalizer of $\mathbf{K}$ in $SymE$. Since an *Automorphism of a permutation group* $(E, \mathbf{G})$ is a permutation $\sigma \in SymE$ with $\sigma \mathbf{G} \sigma^{-1} = \mathbf{G}$ (cf. [11])we have:

$$\mathbf{N} = Aut(E, \mathbf{K}) .$$

If we set "$\nu : SymE \to SymE; \sigma \mapsto \sigma^{-1}$" then by Corollary 3.2, $\Gamma^+(\mathfrak{K})$ is isomorphic to the group $\mathbf{K}^\circ \rtimes i_{\mathbf{N}}$ and $\Gamma(\mathfrak{K})$ to $\mathbf{K}^\circ \rtimes (i_{\mathbf{N}} \oplus \{id, \nu\})$ where $\mathbf{K}^\circ := \{k^\circ \mid k \in \mathbf{K}\}$ and $k^\circ : \mathbf{K} \to \mathbf{K}; x \mapsto k \circ x$. By Theorem 1.10.(2), we have:

Example. $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ is a double symmetric web $\Longleftrightarrow$ $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ is a web satisfying the Reidemeister Condition $\Longleftrightarrow$ $(E, \mathbf{K})$ is a regular permutation group.

If $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ is a double symmetric web, if $o \in E$ is fixed and if for $a \in E$, $a^+ \in \mathbf{K}$ denotes the map uniquely determined by $a^+(o) = a$ then $(E, +)$ with $a + b := a^+(b)$ is a group isomorphic to $(\mathbf{K}, \circ)$ and to $(\mathfrak{K}, \cdot)$. In this case we have $\mathbf{N} = Aut(E, \mathbf{K}) = \mathbf{K} \rtimes Aut(E, +)$ and, by Corollary 3.5, $\Gamma^+(\mathfrak{K})$ is isomorphic to $(\mathbf{K}^\circ \rtimes i_{\mathbf{K}}) \rtimes i_{Aut(E,+)}$ and $\Gamma(\mathfrak{K})$ is isomorphic to $(\mathbf{K}^\circ \rtimes (i_{\mathbf{K}} \oplus \{id, \nu\})) \rtimes i_{Aut(E,+)}$.

Example. $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ is a double symmetric 2-structure $\Longleftrightarrow$ $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ is a 2-structure satisfying the rectangle axiom $\Longleftrightarrow$ $(E, \mathbf{K})$ is a sharply 2-transitive permutation group.

If $(E, \mathbf{K})$ is a sharply 2-transitive permutation group then the set E can be turned into a neardomain $(E, +, *)$ by (cf. [5] § 11):

Let $\mathbf{J} := \{\sigma \in \mathbf{K} \mid \sigma^2 = id \neq \sigma\}$ be the set of all involutory permutations then $\mathbf{J}$ acts semiregularly on E and all elements of $\mathbf{J}$ are conjugate under $\mathbf{K}$. Therefore we can define:

$$char(\mathbf{K}) := 2 \Longleftrightarrow \forall \sigma \in \mathbf{J} : Fix\ \sigma = \emptyset \quad \text{and} \quad char(\mathbf{K}) :\neq 2 \text{ otherwise} .$$

Now let $o, e \in E$ be two distinct fixed elements and let $\mathbf{K}_o := \{\sigma \in \mathbf{K} \mid \sigma(o) = o\}$. If $char(\mathbf{K}) := 2$ let $\mathbf{A} = \mathbf{J} \cup \{id\}$ and if $char(\mathbf{K}) :\neq 2$, let $\omega \in \mathbf{J}$ with $\omega(o) = o$ and $\mathbf{A} := \mathbf{J} \circ \omega$. In both cases the permutation set $\mathbf{A}$ acts regularly on the set E and

therefore if $a \in E$ then there is exactly one element $a^+ \in \mathbf{A}$ such that $a^+(o) = a$. Since $\mathbf{K}_o$ acts regularly on the set $E^* := E \setminus \{o\}$, there is also exactly one element $a^* \in \mathbf{K}_o$ such that $a^*(e) = a$ if $a \neq o$, if $a = o$ we set $o^* := 0$ (the zero map). For $a + b := a^+(b)$ and $a * b := a^*(b)$, $(E, +, *)$ becomes a neardomain.

In this case $(\mathfrak{K}, \cdot)$ and so also $(\mathbf{K}, \circ)$, is isomorphic to the group $T_2(E) := \{\tau_{m,n} |\ m, n \in E,\ n \neq o\}$ where $\tau_{m,n} : x \mapsto m + n * x$. The group $T_2(E)$ can be represented as quasidirect product $T_2(E) = (E, +) \rtimes_Q (E^*, *)$ between the K-loop $(E.+)$ and the group $(E, *)$.

By [11, (1.6), p. 218], the automorphism group $Aut(E, T_2(E))$ of the sharply 2-transitive permutation group $(E, T_2(E))$ is isomorphic to the semidirect product $T_2(E) \rtimes Aut(E, +, *)$. Thus we have $\mathbf{N} = T_2(E) \rtimes Aut(E, +, *)$ and, by Corollary 3.5, $\Gamma^+(\mathfrak{K})$ is isomorphic to $(\mathbf{K}^\circ \rtimes i_{\mathbf{K}}) \rtimes i_{Aut(E,+,*)}$ and $\Gamma(\mathfrak{K})$ is isomorphic to $(\mathbf{K}^\circ \rtimes (i_{\mathbf{K}} \oplus \{id, \nu\})) \rtimes i_{Aut(E,+,*)}$.

Example. Let $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ be a hyperbola structure satisfying the rectangle axiom. Then $(E, \mathbf{K})$ with $\mathbf{K} := \mathbf{\Pi_E}(\mathfrak{K})$ is a subgroup of $Sym\ E$ acting sharply 3-transitive on E. If one chooses an element $\infty \in E$ and puts $F := E \setminus \{\infty\}$ then the stabilizer $\mathbf{K}_\infty := \{\sigma \in \mathbf{K} \mid \sigma(\infty) = \infty\}$ of ∞ acts sharply 2-transitive on F and so - like in Ex 2 - F can be turned in a neardomain $(F, +, *)$. There is exactly one involutory permutation $\varepsilon \in \mathbf{K}$ with $\varepsilon(o) = \infty$ and $\varepsilon(e) = e =: 1$ and the restriction of ε onto $F^* := F \setminus \{o\}$ is an involutory automorphism of the group $(F^*, *)$ satisfying the functional equation

$$\varepsilon\big(1 - \varepsilon(x)\big) = 1 - \varepsilon(1 - x) \quad \text{for all} \quad x \in F \setminus \{0, 1\}\,.$$

Such a structure $(F, +, *, \varepsilon)$ is called a *KT-field*.

Now let T_2 be the sharply 2-transitive permutation group of the neardomain $(F, +, *)$ according to Example 2 where the maps $\tau \in T_2(F, +, *)$ are extended on E by $\tau(\infty) = \infty$ then $T_2 = \mathbf{K}_\infty$ and $\mathbf{K} = T_3(F, \varepsilon) := T_2 \cup (T_2 \circ \varepsilon \circ T_2)$(cf. [12], [11, (3.1), p. 235]).

Conversely if $(F, +, *, \varepsilon)$ is a KT-field, if the set F is extended by an element ∞ to $E := F \cup \{\infty\}$ and if additionally we put $\varepsilon(\infty) = 0$, $\varepsilon(0) = \infty$ and $\tau(\infty) = \infty$ for $\tau \in T_2(F, +, *)$ then $(E, T_3(F, \varepsilon))$ is a sharply 3-transitive permutation group (cf. [11, (3.3), p. 236]).

Again, by [11, (3.4.(b)), p. 237]), $Aut(E, T_3(F, \varepsilon)) = T_3(F, \varepsilon) \rtimes Aut(F, \varepsilon)$. Thus we have $\mathbf{N} = T_3(F, \varepsilon) \rtimes Aut(F, \varepsilon)$ and, by Corollary 3.5, $\Gamma^+(\mathfrak{K})$ is isomorphic to $(\mathbf{K}^\circ \rtimes i_{\mathbf{K}}) \rtimes i_{Aut(F,\varepsilon)}$ and $\Gamma(\mathfrak{K})$ is isomorphic to $(\mathbf{K}^\circ \rtimes (i_{\mathbf{K}} \oplus \{id, \nu\})) \rtimes i_{Aut(F,\varepsilon)}$.

4. Automorphisms of symmetric chain structures

In this section let $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ be a symmetric chain structure, let $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{C})$ be the corresponding maximal chain structure, let $E \in \mathfrak{K}$ be fixed, let $(\mathfrak{C}, \cdot) := (\mathfrak{C}, \cdot\,; E)$ and let $\overline{\mathfrak{K}} := \langle \mathfrak{K} \rangle$ be the subgroup of $(\mathfrak{C}, \cdot)$ generated by $\mathfrak{K}$.

Proposition 4.1. *For each $K \in \mathfrak{K}$ let $\langle K \rangle$ be the subgroup of $(\mathfrak{C}, \cdot)$ generated by K and let $\widetilde{\mathfrak{K}}(E) := \{A_n \cdot \cdots \cdot A_1 \cdot A_1 \cdots \cdot A_n \mid n \in \mathbb{N} : A_1, \ldots, A_n \in \mathfrak{K}\}$ then:*

1. $\mathfrak{K}^{-1} = \mathfrak{K}$, $\forall K \in \mathfrak{K} : K \cdot \mathfrak{K} \cdot K = \mathfrak{K}$ *and* $\langle K \rangle \subseteq \mathfrak{K}$.
2. $\widetilde{\mathfrak{K}}(E) \subset \mathfrak{K}$.
3. $\forall X \in \overline{\mathfrak{K}}\ \exists\ K_1, K_2, \ldots, K_n \in \mathfrak{K} : X = K_1 \cdot K_2 \cdots K_n$.
4. $\forall A, B \in \mathfrak{K} : \widetilde{A}(B) = \widetilde{A} \circ \widetilde{B} \circ \widetilde{A}$.

Proof. (1). Since $E \in \mathfrak{K}$, $K^{-1} = \widetilde{E}(K) \in \mathfrak{K}$ and so $\mathfrak{K}^{-1} = \mathfrak{K}$ implying $K \cdot \mathfrak{K} \cdot K = \mathfrak{K}$. Consequently, $K^2 = K \cdot E \cdot K \in \mathfrak{K}$, $K^3 = K \cdot K \cdot K \in \mathfrak{K}$ and by induction, $K^n \in \mathfrak{K}$ for all $n \in \mathbb{N}$. This shows $\langle K \rangle \subseteq \mathfrak{K}$ and moreover the validity of statement (2). (3) is a consequence of (1). □

Definition 4.2. For each $C \in \mathfrak{C}$, $K \in \mathfrak{K}$ let:

1. $\mathfrak{K}_C := \{K \in \mathfrak{K} \mid C \cdot \mathfrak{K} \cdot C^{-1} \cdot K = \mathfrak{K}\}$,
2. $K_{\mathfrak{C}} := \{C \in \mathfrak{C} \mid C \cdot \mathfrak{K} \cdot C^{-1} \cdot K = \mathfrak{K}\}$,
3. $\mathfrak{K}_{\mathfrak{C}} := \bigcup\{\mathfrak{K}_C \mid C \in \mathfrak{C} : \mathfrak{K}_C \neq \emptyset\}$,
4. $\mathfrak{C}_{\mathfrak{K}} := \{C \in \mathfrak{C} \mid \mathfrak{K}_C \neq \emptyset\}$,
5. $\mathbf{s}(\mathfrak{K}) := \{K \in \mathfrak{K} |\ K \cdot \mathfrak{K} = \mathfrak{K}\}$.

Proposition 4.3. $\mathbf{s}(\mathfrak{K}) \cdot \mathfrak{K} = \mathfrak{K} \cdot \mathbf{s}(\mathfrak{K}) = \mathfrak{K}$ *hence* $\mathbf{s}(\mathfrak{K})$ *is a normal subgroup of* $\mathfrak{N}_{\mathfrak{C}}(\mathfrak{K})$ *contained in* $\mathfrak{K}$.

Proof. For any $K \in \mathfrak{K}$ we have by (4.1), $K \cdot \mathfrak{K} \cdot K = \mathfrak{K}$. Thus for $S \in \mathbf{s}(\mathfrak{K})$ we have $\mathfrak{K} = S^{-1} \cdot \mathfrak{K} = S^{-1} \cdot S \cdot \mathfrak{K} \cdot S = \mathfrak{K} \cdot K$, hence $S \cdot \mathfrak{K} = \mathfrak{K} = \mathfrak{K} \cdot S$, i.e. $S \in \mathfrak{N}_{\mathfrak{C}}(\mathfrak{K})$. □

Theorem 4.4. $\mathfrak{K} \subseteq \overline{\mathfrak{K}} := \langle \mathfrak{K} \rangle \leq \mathfrak{C}_{\mathfrak{K}} \leq (\mathfrak{C}, \cdot)$ *and*

1. $\Gamma^-(\mathfrak{K}) = \{C(\widetilde{C^{-1} \cdot K}) \mid C \in \mathfrak{C}_{\mathfrak{K}},\ K \in \mathfrak{K}_C\}$,
2. $\Gamma^+(\mathfrak{K}) = \{C(\widetilde{C^{-1} \cdot K}) \circ \widetilde{E} \mid C \in \mathfrak{C}_{\mathfrak{K}},\ K \in \mathfrak{K}_C\}$,
3. $\Gamma_1(\mathfrak{K}) = \{\widetilde{AE} \circ \widetilde{E} |\ A \in \mathbf{s}(\mathfrak{K})\}$,
4. $\Gamma_2(\mathfrak{K}) = \{\widetilde{EA} \circ \widetilde{E} |\ A \in \mathbf{s}(\mathfrak{K})\}$.

Proof. Let $C, D \in \mathfrak{C}_{\mathfrak{K}}$ and $K_C, K_D \in \mathfrak{K}$ with $C \cdot \mathfrak{K} \cdot C^{-1} \cdot K_C = D \cdot \mathfrak{K} \cdot D^{-1} \cdot K_D = \mathfrak{K}$. Then $\mathfrak{K} = D \cdot C \cdot \mathfrak{K} \cdot C^{-1} \cdot K_C \cdot D^{-1} \cdot K_D = D \cdot C \cdot \mathfrak{K} \cdot C^{-1} \cdot D^{-1} \cdot D \cdot K_C \cdot D^{-1} \cdot K_D$ and $D \cdot K_C \cdot D^{-1} \cdot K_D \in D \cdot \mathfrak{K} \cdot D^{-1} \cdot K_D = \mathfrak{K}$ hence $D \cdot C \in \mathfrak{C}_{\mathfrak{K}}$. Moreover $\mathfrak{K} = C^{-1} \cdot \mathfrak{K} \cdot K_C^{-1} \cdot C = C^{-1} \cdot \mathfrak{K} \cdot C \cdot C^{-1} \cdot K_C^{-1} \cdot C$ and $C^{-1} \cdot K_C^{-1} \cdot C = C^{-1} \cdot E \cdot K_C^{-1} \cdot C \in C^{-1} \cdot \mathfrak{K} \cdot K_C^{-1} \cdot C = \mathfrak{K}$, i.e. $C^{-1} \in \mathfrak{C}_{\mathfrak{K}}$. Thus $\mathfrak{C}_{\mathfrak{K}} \leq (\mathfrak{C}, \cdot)$. If $K \in \mathfrak{K}$ then by 4.1., $K^2 \in \mathfrak{K}$ and $\mathfrak{K} = K \cdot \mathfrak{K} \cdot K = K \cdot \mathfrak{K} \cdot K^{-1} \cdot K^2$ hence $K^2 \in \mathfrak{K}_K$, i.e. $K \in \mathfrak{C}_{\mathfrak{K}}$ and so $\mathfrak{K} \subseteq \mathfrak{C}_{\mathfrak{K}}$.

(1) Let $\widetilde{AB} \in \Gamma^-(\mathfrak{K})$. Then by 2.1.(5), $A \cdot \mathfrak{K} \cdot B = \mathfrak{K}^{-1} = \mathfrak{K}$ hence $A \cdot B = A \cdot E \cdot B =: K \in \mathfrak{K}$, i.e. $B = A^{-1} \cdot K$ and $\widetilde{AB} = A(\widetilde{A^{-1} \cdot K})$ with $K \in \mathfrak{K}_A$ and so $A \in \mathfrak{C}_{\mathfrak{K}}$.

(4) $\widetilde{EA} \circ \widetilde{E}(\mathfrak{K}) = \mathfrak{K} \Leftrightarrow \mathfrak{K} \cdot A = \mathfrak{K} \Leftrightarrow A \in \mathbf{s}(\mathfrak{K})$.

(2) follows from (1) and 4.1.(1). By 4.3., (3) follows analogous to the proof of (4). □

Corollary 4.5. $\Gamma(\mathfrak{K}) \leq \Gamma(\overline{\mathfrak{K}})$.

Proof. Let $A, B \in \mathfrak{K},\ C \in \mathfrak{C}_{\mathfrak{K}},\ K \in \mathfrak{K}_C$ then $C(\widetilde{C^{-1}\cdot K}) \circ \widetilde{E}(A\cdot B) = C\cdot A\cdot B\cdot C^{-1}\cdot K = (C\cdot A\cdot C^{-1}\cdot K)\cdot K^{-1}\cdot(C\cdot B\cdot C^{-1}\cdot K) \in \mathfrak{K}\cdot\mathfrak{K}\cdot\mathfrak{K} \subseteq \overline{\mathfrak{K}}$. Moreover if $A_1, A_2, \ldots, A_n \in \mathfrak{K}$ then $\widetilde{E}(A_1\cdot A_2\cdot\cdots\cdot A_n) = A_n^{-1}\cdot\cdots\cdot A_2^{-1}\cdot A_1^{-1} \in \overline{\mathfrak{K}}$. □

Using Theorem 4.4. we can rewrite Theorem 1.12.(4) in a stronger form:

Corollary 4.6. $\mathfrak{K}^s \subseteq \mathfrak{C}_{\mathfrak{K}} \cap \{A \in \mathfrak{C} |\ A^2 \in \mathfrak{K}_{\mathfrak{C}}\}$.

Proof. By Theorem 4.4(1), since $E \in \mathfrak{K} = \mathfrak{K}^{-1}$, if $\widetilde{A}(\mathfrak{K}) = A\cdot\mathfrak{K}^{-1}\cdot A = \mathfrak{K}$ then $K := A^2 \in \mathfrak{K}$ and so $A\cdot\mathfrak{K}\cdot A^{-1}\cdot K = A\cdot\mathfrak{K}\cdot A = \mathfrak{K}$, i.e. $K \in \mathfrak{K}_A$ and so $A \in \mathfrak{C}_{\mathfrak{K}}$. □

Theorem 4.7. $\mathbf{s}(\mathfrak{K}) \trianglelefteq \mathfrak{N}_{\mathfrak{C}}(\mathfrak{K}) \leq \mathfrak{C}_{\mathfrak{K}} \leq \mathfrak{N}_{\mathfrak{C}}(\overline{\mathfrak{K}})$.

Proof. By 4.3., $\mathbf{s}(\mathfrak{K}) \leq \mathfrak{N}_{\mathfrak{C}}(\mathfrak{K})$. Let $C \in \mathfrak{N}_{\mathfrak{C}}(\mathfrak{K})$ then $E \in \mathfrak{K}_C$ hence $C \in \mathfrak{C}_{\mathfrak{K}}$, i.e. $\mathfrak{N}_{\mathfrak{C}}(\mathfrak{K}) \subseteq \mathfrak{C}_{\mathfrak{K}}$ and by 4.4., $\mathfrak{N}_{\mathfrak{C}}(\mathfrak{K}) \leq \mathfrak{C}_{\mathfrak{K}}$. For any $C \in \mathfrak{C}_{\mathfrak{K}}$ there is a $K \in \mathfrak{K}_C \subseteq \mathfrak{K}$ such that $C\cdot\mathfrak{K}\cdot C^{-1} = \mathfrak{K}\cdot K^{-1} \subseteq \mathfrak{K}\cdot\mathfrak{K} \subset \overline{\mathfrak{K}}$ implying $\mathfrak{C}_{\mathfrak{K}} \leq \mathfrak{N}_{\mathfrak{C}}(\overline{\mathfrak{K}})$. □

Theorem 4.8.

1. *If $K \in \mathfrak{K}_{\mathfrak{C}}$, then $K_{\mathfrak{C}} = C\cdot\mathfrak{N}_{\mathfrak{C}}(\mathfrak{K})$ for any $C \in K_{\mathfrak{C}}$ hence $K_{\mathfrak{C}} = K_{\mathfrak{C}}\cdot\mathfrak{N}_{\mathfrak{C}}(\mathfrak{K})$.*
2. *If $C \in \mathfrak{C}_{\mathfrak{K}}$, then $\mathfrak{K}_C = K\cdot\mathbf{s}(\mathfrak{K})$ for any $K \in \mathfrak{K}_C$ hence $\mathfrak{K}_C = \mathfrak{K}_C\cdot\mathbf{s}(\mathfrak{K})$.*

Proof. (1) For $C, D \in K_{\mathfrak{C}}$ we have: $\mathfrak{K}\cdot K^{-1} = C\cdot\mathfrak{K}\cdot C^{-1} = D\cdot\mathfrak{K}\cdot D^{-1}$, hence $C^{-1}\cdot D \in \mathfrak{N}_{\mathfrak{C}}(\mathfrak{K})$, i.e. $D \in C\cdot\mathfrak{N}_{\mathfrak{C}}(\mathfrak{K})$ and so $K_{\mathfrak{C}} \subseteq C\cdot\mathfrak{N}_{\mathfrak{C}}(\mathfrak{K})$. If $N \in \mathfrak{N}_{\mathfrak{C}}(\mathfrak{K})$ and $C \in K_{\mathfrak{C}}$ hence $N\cdot\mathfrak{K}\cdot N^{-1} = \mathfrak{K}$ and $C\cdot\mathfrak{K}\cdot C^{-1}\cdot K = \mathfrak{K}$ then $C\cdot N\cdot\mathfrak{K}\cdot N^{-1}\cdot C^{-1}\cdot K = C\cdot\mathfrak{K}\cdot C^{-1}\cdot K = \mathfrak{K}$ hence $C\cdot N \in K_{\mathfrak{C}}$.

(2) For $K, L \in \mathfrak{K}_C$ we have: $\mathfrak{K}\cdot K^{-1} = \mathfrak{K}\cdot L^{-1} = C\cdot\mathfrak{K}\cdot C^{-1}$, hence by $\mathfrak{K}^{-1} = \mathfrak{K}$, $\mathfrak{K} = \mathfrak{K}\cdot L^{-1}\cdot K = K^{-1}\cdot L\cdot\mathfrak{K}$, i.e. $K^{-1}\cdot L \in \mathbf{s}(\mathfrak{K})$. □

Theorem 4.9. $\widetilde{\mathfrak{K}}(E) \subset \mathfrak{K}_{\mathfrak{C}}$ *and* $\bigcup_{C\in\overline{\mathfrak{K}}}\mathfrak{K}_C = \widetilde{\mathfrak{K}}(E)\cdot\mathbf{s}(\mathfrak{K})$.

Proof. Let $K := A_1\cdots A_n\cdot A_n\cdots A_1 \in \widetilde{\mathfrak{K}}(E)$ with $A_1, \ldots, A_n \in \mathfrak{K}$ (cf. 4.1.(3)) and let $C := A_1\cdots A_n$. Then $C \in \overline{\mathfrak{K}}$, by 4.1.(1), $C\cdot\mathfrak{K}\cdot C^{-1}\cdot K = A_1\cdot A_2\cdots A_n\cdot\mathfrak{K}\cdot A_n\cdots A_1 = \mathfrak{K}$, i.e. $K \in \mathfrak{K}_C \subseteq \mathfrak{K}_{\mathfrak{C}}$ and $C \in \mathfrak{C}_{\mathfrak{K}}$ hence $\widetilde{\mathfrak{K}}(E) \subseteq \bigcup_{C\in\overline{\mathfrak{K}}}\mathfrak{K}_C \subseteq \mathfrak{K}_{\mathfrak{C}}$. Therefore by 4.8.(2), $\widetilde{\mathfrak{K}}(E)\cdot\mathbf{s}(\mathfrak{K}) \subseteq \bigcup_{C\in\overline{\mathfrak{K}}}\mathfrak{K}_C$.

Now let $X \in \bigcup_{C\in\overline{\mathfrak{K}}}\mathfrak{K}_C$, i.e. $\exists C \in \overline{\mathfrak{K}} : X \in \mathfrak{K}_C$. Then $\exists A_1, \ldots, A_n \in \mathfrak{K} : C = A_1\cdots A_n$, $C\cdot\mathfrak{K}\cdot C^{-1}\cdot X = \mathfrak{K}$ and $D := C\cdot A_n\cdots A_1 \in \widetilde{\mathfrak{K}}(E)$. Therefore $\mathfrak{K} = C\cdot\mathfrak{K}\cdot C^{-1}\cdot X = A_1\cdots A_n\cdot\mathfrak{K}\cdot A_n\cdots A_1\cdot D^{-1}\cdot X = \mathfrak{K}\cdot D^{-1}\cdot X$ implying $S := D^{-1}\cdot X \in \mathfrak{K}$ and by 4.3., $S \in \mathbf{s}(\mathfrak{K})$. Thus $X = D\cdot S \in \widetilde{\mathfrak{K}}(E)\cdot\mathbf{s}(\mathfrak{K})$. □

Corollary 4.10. $\mathfrak{C}_{\mathfrak{K}} = \langle\mathfrak{N}_{\mathfrak{C}}(\mathfrak{K}) \cup \overline{\mathfrak{K}}\rangle \Leftrightarrow \mathfrak{K}_{\mathfrak{C}} = \widetilde{\mathfrak{K}}(E)\cdot\mathbf{s}(\mathfrak{K})$. *In this case we have:*

$$\Gamma(\mathfrak{K}) = \left\langle \widetilde{\mathfrak{K}} \cup \left\{\widetilde{CC^{-1}} |\ C \in \mathfrak{N}_{\mathfrak{C}}(\mathfrak{K})\right\} \cup \left\{\widetilde{SE} |\ S \in \mathbf{s}(\mathfrak{K})\right\}\right\rangle.$$

Corollary 4.11. *If $\mathfrak{C}_{\mathfrak{K}} = \mathfrak{N}_{\mathfrak{C}}(\mathfrak{K})$, then $\Gamma(\mathfrak{K}) = \langle\{\widetilde{CC^{-1}} |\ C \in \mathfrak{N}_{\mathfrak{C}}(\mathfrak{K})\} \cup \{\widetilde{SE} |\ S \in \mathbf{s}(\mathfrak{K})\}\rangle$, and any anti-automorphism is a composition of an anti-automorphisms from stabilizer of E with an automorphism from $\Gamma_2(\mathfrak{K})$,*

$$C(\widetilde{C^{-1}\cdot K_C}) = (\widetilde{EK_C} \circ \widetilde{E}) \circ \widetilde{CC^{-1}}.$$

Proposition 4.12. $\mathfrak{N}_{\mathfrak{C}}(\mathfrak{K}) \cap \mathfrak{K} = \{A \in \mathfrak{K} |\ A^2 \in \mathbf{s}(\mathfrak{K})\}$.

Proof. Let $A \in \mathfrak{N}_{\mathfrak{C}}(\mathfrak{K}) \cap \mathfrak{K}$ then $\mathfrak{K} = A \cdot \mathfrak{K} \cdot A^{-1} = A^2 \cdot (A^{-1} \cdot \mathfrak{K} \cdot A^{-1}) = A^2 \cdot \mathfrak{K}$, i.e. $A \in \mathfrak{K}$ and $A^2 \in \mathbf{s}(\mathfrak{K})$. Conversely if $A \in \mathfrak{K}$ with $A^2 \in \mathbf{s}(\mathfrak{K})$ then $\mathfrak{K} = A^2 \cdot \mathfrak{K} = A \cdot (A \cdot \mathfrak{K} \cdot A) \cdot A^{-1} = A \cdot \mathfrak{K} \cdot A^{-1}$, i.e. $A \in \mathfrak{N}_{\mathfrak{C}}(\mathfrak{K}) \cap \mathfrak{K}$. □

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Received: July 1, 2009.
Revised: August 18, 2009.