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A Characterization of Projective Spaces by a Set of Planes

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Abstract. This note deals with the following question: How many planes of a linear space $(P, \mathfrak{L})$ must be known as projective planes to ensure that $(P, \mathfrak{L})$ is a projective space? The following answer is given: If for any subset M of a linear space $(P, \mathfrak{L})$ the restriction $(M, \mathfrak{L}(M))$ is locally complete, and if for every plane E of $(M, \mathfrak{L}(M))$ the plane $\overline{E}$ generated by E is a projective plane, then $(P, \mathfrak{L})$ is a projective space. Or more generally: If for any subset M of P the restriction $(M, \mathfrak{L}(M))$ is locally complete, and if for any two distinct coplanar lines $G_1, G_2 \in \mathfrak{L}(M)$ the lines $\overline{G_1}, \overline{G_2} \in \mathfrak{L}$ generated by G_1, G_2 have a nonempty intersection and $\overline{G_1 \cup G_2}$ satisfies the exchange condition, then $(P, \mathfrak{L})$ is a generalized projective space.

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1. Introduction

For a subset $M \subset P$ of a linear space $(P, \mathfrak{L})$ let $\mathfrak{L}(M) := \{L \cap M : L \in \mathfrak{L}$ and $|L \cap M| \geqslant 2\}$. We call the linear space $(M, \mathfrak{L}(M))$ a *restriction* of $(P, \mathfrak{L})$, i.e., the lines of $(M, \mathfrak{L}(M))$ are exactly parts of the lines of $\mathfrak{L}$.

An *embedding* ϕ: $M \to P$ of a linear space $(M, \mathfrak{M})$ in a linear space $(P, \mathfrak{L})$ is an injective mapping which maps collinear points onto collinear points and noncollinear points onto noncollinear points. For an embedding ϕ: $M \to P$ we have $\phi(\mathfrak{M}) := \{\phi(G) : G \in \mathfrak{M}\} = \mathfrak{L}(\phi(M))$; this means, $(\phi(M), \phi(\mathfrak{M}))$ is a restriction of $(P, \mathfrak{L})$. Usually we identify M and $\phi(M)$ for an embedding ϕ, hence $(\phi(M), \phi(\mathfrak{M})) = (M, \mathfrak{L}(M))$.

We call a restriction $(M, \mathfrak{L}(M))$ of $(P, \mathfrak{L})$ *locally complete*, if for every nonempty subspace T of $(M, \mathfrak{L}(M))$ there exists exactly one subspace U of $(P, \mathfrak{L})$ with $T = M \cap U$ (cf. [4, 8]).

For a restriction M of $(P, \mathfrak{L})$ we consider in this paper the following property that for every plane E of M the plane of P generated by E is a projective plane (property (P1) of Section 3), or as a generalization, that for any two coplanar lines G_1, G_2 of E the lines of P generated by G_1, G_2 have a nonempty intersection (property (P4) of Section 3).

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In this paper it is proved that for every locally complete restriction M of $(P, \mathfrak{L})$ satisfying (P1), or one of the equivalent properties (P2) or (P3) (cf. Section 3) the linear space $(P, \mathfrak{L})$ is a generalized projective space, and $(M, \mathfrak{L}(M))$ is locally generalized projective and satisfies the Bundle Theorem.

Thereby a linear space $(M, \mathfrak{M})$ is called *locally (generalized) projective,* if for every point $x \in M$ the lines and planes of $(M, \mathfrak{M})$ containing x form a (generalized) projective space. We recall a result of O. Wyler [10, Theorem 6.2] and L. M. Batten [1]:

LEMMA 1.1. *A locally generalized projective space is a locally projective space or a generalized projective space.*

The Bundle Theorem states that for four lines $A, B, C, D \in \mathfrak{M}$, no three in a common plane, the coplanarities of $\{A, B\}, \{A, C\}, \{A, D\}, \{B, C\}, \{B, D\}$ imply the coplanarity of $\{C, D\}$.

There is a close connection of the result mentioned above to the classical Embedding Theorems of O. Wyler [10], W. M. Kantor [3], J. Kahn and many others, which state that every locally generalized projective linear space $(M, \mathfrak{M})$ of dim $M \geqslant 3$ satisfying the Bundle Theorem is embeddable in a generalized projective space. For dim $M = 3$ the result is due to J. Kahn [2] (cf. also [5]). For these embeddings the so-called *bundle space* $(P', \mathfrak{L}')$ is constructed. The points of P' are defined by *bundles* of $\mathfrak{b} \subset \mathfrak{M}$, whereby any two lines of a bundle are coplanar and every point of M is incident with a line of every bundle (cf. [8]). For $y \in M$ let $[y] := \{G \in \mathfrak{M} : y \in G\}$. For any two bundles $\mathfrak{a}, \mathfrak{b}$ of the bundle space the connecting line $[\mathfrak{a}, \mathfrak{b}]$ consists of all bundles $\mathfrak{x}$, for which the unique lines of $\mathfrak{a}, \mathfrak{b}, \mathfrak{x}$ through every point $y \in M$ with $[y] \neq \mathfrak{a}, \mathfrak{b}, \mathfrak{x}$ are coplanar.

The mapping ϕ: $M \to P', x \mapsto [x]$ embeds M in its bundle space P'. It is easy to show that $\phi(M)$ is a locally complete restriction of P. Hence we have a situation similar to that of this paper.

We want to mention the paper [9] in which it is proved that a linear space is an affine or projective space, if every plane is an affine or projective plane.

2. The Dimension of a Locally Complete Restriction

Let $(P, \mathfrak{L})$ denote a *linear space* with the point set P and the line set $\mathfrak{L}$. A *subspace* is a subset $U \subset P$ such that for all distinct points $x, y \in U$ the unique line passing through x and y, denoted by $\overline{x, y}$, is contained in U. Let $\mathfrak{U}$ denote the set of all subspaces. For every subset $X \subset P$ we define the following *closure operation*

$$\overline{} : \mathfrak{P}(P) \to \mathfrak{U}, \qquad X \mapsto \overline{X} \quad \text{by} \quad \overline{X} := \bigcap_{\substack{U \in \mathfrak{U} \\ X \subset U}} U \tag{1}$$

For $U \in \mathfrak{U}$ we call dim $U := \inf\{|X| - 1 : X \subset U$ and $\overline{X} = U\}$ the *dimension* of U. A subspace of dimension two is a *plane*. A subset $X \subset P$ is *independent,* if

$x \notin \overline{X \setminus \{x\}}$ for every $x \in X$, and is a *basis* of a subspace U, if X is independent and $\overline{X} = U$.

A linear space is a *generalized projective plane*, if any two lines have a nonempty intersection, and a *projective plane*, if in addition every line contains at least three points. A linear space is a *(generalized) projective space*, if every plane is a (generalized) projective plane. A generalized projective space can be defined equivalently by the Axiom of Veblen–Young.

A linear space $(P, \mathfrak{L})$ satisfies the *exchange condition* if for $S \subset P$ and $x, y \in P$ with $x \in \overline{S \cup \{y\}} \setminus \overline{S}$, it follows that $y \in \overline{S \cup \{x\}}$.

For $M \subset P$, let $(M, \mathfrak{L}(M))$ be always a restriction of the linear space $(P, \mathfrak{L})$. We denote by $X \mapsto \overline{X}$ the closure of $(P, \mathfrak{L})$ and by $X \mapsto \langle X \rangle$ the closure of $(M, \mathfrak{L}(M))$. By [4, (1.1)] and [6, Lemma 2.3]:

LEMMA 2.1. (1) *For every subspace U of $(P, \mathfrak{L})$, $U \cap M$ is a subspace of $(M, \mathfrak{L}(M))$.*

(2) *For a subset $X \subset M$ it holds that $\langle X \rangle \subset \overline{X}$ and $\overline{\langle X \rangle} = \overline{X}$.*

(3) *If a subset $X \subset M$ is independent in P, then X is also independent in M.*

(4) *If $\overline{M} = P$, then* $\dim M \geqslant \dim P$.

By [4, (1.5)]:

LEMMA 2.2. *For a restriction $(M, \mathfrak{L}(M))$ of $(P, \mathfrak{L})$, the following statements are equivalent:*

(1) *$(M, \mathfrak{L}(M))$ is locally complete.*
(2) *For every subspace T of M and for every subspace U of P with $U \cap M \neq \emptyset$ it holds:*

$$U = \overline{U \cap M} \qquad \textit{and} \qquad T = \overline{T} \cap M. \tag{2}$$

(3) *The properties* (**G**) *and* (**E**) *are both satisfied, whereby*

(G) *For every line $L \in \mathfrak{L}$, $|L \cap M| \neq 1$.*
(E) *For every plane E of M, $E = \overline{E} \cap M$.*

By Lemma 2.2(2) $P = \overline{P \cap M} = \overline{M}$. Hence:

LEMMA 2.3. *If $(M, \mathfrak{L}(M))$ is a locally complete restriction of $(P, \mathfrak{L})$, then $\overline{M} = P$.*

THEOREM 2.4. *Let $(M, \mathfrak{L}(M))$ be a locally complete restriction of $(P, \mathfrak{L})$. Then* $\dim P + 1 \geqslant \dim M \geqslant \dim P$.

Proof. By 2.3 and 2.1(4) $\dim M \geqslant \dim P$. Let $X \subset P$ be a generating subset of P and $a \in M \setminus X$. By (G) there exists for every $x \in X$ a point $b_x \in \overline{x, a} \cap M$ with $b_x \neq a$, hence also $B := \{b_x : x \in X\} \cup \{a\}$ is a generating subset of P

with $B \subset M$ and $|B| = |X| + 1$. Hence by 2.1(2), $\langle B\rangle = \overline{\langle B\rangle} \cap M = \overline{B} \cap M = P \cap M = M$, since M is a locally complete restriction. Therefore B is a generating subset of M and we have $\dim M \leqslant 1 + \dim P$. □

Remark 1. We remark that property (E) cannot be omitted in Theorem 2.4. In [7] an example of an embedding of a high-dimensional projective space in a Desarguesian projective plane satisfying (G) is given (which by 2.4 cannot satisfy (E))

In [6] it is shown that for any $n, s \in \mathbb{N}, n \geqslant 2$, every Pappian projective space $(M, \mathfrak{M})$ of $\dim M = n + s$ is embeddable in a Pappian projective space $(P, \mathfrak{L})$ of $\dim P = n$, i.e., we can find every Pappian projective space M of $\dim M = n+s$ as a restriction of an n-dimensional Pappian projective space $(P, \mathfrak{L})$. The restriction has the property that for every subspace T of M of $\dim T \leqslant n - 1$ it holds that $T = \overline{T} \cap M$. In particular for $n \geqslant 2$, (E) is satisfied (but not (G)).

We recall (cf. [6, Lemma 2.5]:

LEMMA 2.5. *If $(M, \mathfrak{L}(M))$ is a locally complete restriction of a linear space $(P, \mathfrak{L})$ satisfying the exchange condition, then* $\dim M = \dim P$.

3. The Planes of a Locally Complete Restriction

For a restriction M of $(P, \mathfrak{L})$ we consider now the following properties:

(P1) For every plane E of $(M, \mathfrak{L}(M))$, $\overline{E}$ is a generalized projective plane.

(P2) For every plane E of $(M, \mathfrak{L}(M))$ and any lines $G \subset E, L \subset \overline{E}$ it holds $\overline{G} \cap L \neq \emptyset$.

(P3) For every plane E of $(M, \mathfrak{L}(M))$ and any lines $G_1, G_2 \subset E$, it holds $\overline{G}_1 \cap \overline{G}_2 \neq \emptyset$, and $\overline{E}$ satisfies the exchange condition.

(P4) For every plane E of $(M, \mathfrak{L}(M))$ and any lines $G_1, G_2 \subset E$, it holds $\overline{G}_1 \cap \overline{G}_2 \neq \emptyset$.

Clearly (P1) implies (P2) and (P3), and (P4) is a consequence of every (Pi), $i = 1, 2, 3$.

For $M \subset P$, let $(M, \mathfrak{L}(M))$ be in the following a locally complete restriction of the linear space $(P, \mathfrak{L})$ satisfying (P4). We recall that $X \mapsto \overline{X}$ denotes the closure of $(P, \mathfrak{L})$ and $X \mapsto \langle X\rangle$ the closure of $(M, \mathfrak{L}(M))$. We shall abuse the notation when the set is listed $\{a, b, c, \ldots\}$ and write $\overline{a, b, c, \ldots}$ and $\langle a, b, c, \ldots\rangle$.

LEMMA 3.1. (1) *If $a, b, c \in M$ are not collinear points, then $\overline{a, b, c} = \bigcup_{x\in\overline{b,c}} \overline{a, x}$ and $\langle a, b, c\rangle = \bigcup_{x\in\overline{b,c}} \overline{a, x} \cap M$.*

(2). *Every plane E of M satisfies the exchange condition.*

Proof. (1) Since $a, b, c \in \overline{a, b, c}$ and since for any point $x \in \overline{a, b, c}$ it holds $\overline{a, x} \subset \overline{a, b, c}$ we have $\bigcup_{x\in\overline{b,c}} \overline{a, x} \subset \overline{a, b, c}$. Let $y \in \overline{a, b, c} \setminus \{a\}$, then $\overline{a, y} \cap$

$M \subset \langle a, b, c\rangle$ by (E) and (G), hence $\overline{a, y} \cap M$ and $\langle b, c\rangle$ are coplanar in M. By (P4) $p = \overline{b, c} \cap \overline{a, y}$ exists and $\overline{a, y} = \overline{a, p}$, hence $y \in \overline{a, p} \subset \bigcup_{x\in\overline{b,c}} \overline{a, x}$ and $\overline{a, b, c} \subset \bigcup_{x\in\overline{b,c}} \overline{a, x}$. By (E), $\langle a, b, c\rangle = \overline{a, b, c} \cap M = \bigcup_{x\in\overline{b,c}} \overline{a, x} \cap M$.

(2) There are $a, b, c \in E$ with $E = \langle a, b, c\rangle$. Let $d \in E \setminus \langle b, c\rangle$, then by (1) there is an $x \in \langle b, c\rangle$ with $d \in \langle a, x\rangle$, hence $a \in \langle d, x\rangle \subset \langle b, c, d\rangle$ and $E = \langle b, c, d\rangle$. □

LEMMA 3.2. *Let $E, F \subset M$ be distinct planes of M containing distinct common points $x, y \in \overline{E} \cap \overline{F}$. Then for every line $G \subset E$ it holds $\overline{G} \cap \overline{F} \neq \emptyset$.*

Proof. If $x \in M$ or $y \in M$, then $\overline{x, y} \cap M \in \mathfrak{L}(M)$ by (G). Hence (P4) implies $\overline{G} \cap \overline{x, y} \neq \emptyset$ and therefore $\overline{G} \cap \overline{F} \neq \emptyset$. Now we assume $x, y \notin M$. By (P4) for distinct points $a, b \in G$ there exists $z = \overline{a, x} \cap \overline{b, y}$. Let $c \in F \setminus E$ and $d \in (\overline{c, x} \cap M) \setminus \{c\}$.

By (E) and (G), $\langle a, d\rangle$, $\overline{c, z} \cap M \subset \overline{a, c, x}$ are coplanar in M, hence $v = \overline{a, d} \cap \overline{c, z}$ exists. Again by (E) and (G), $\overline{b, v} \cap M, \overline{c, y} \cap M \subset \overline{b, c, y}$ are coplanar in M, hence $w = \overline{b, v} \cap \overline{c, y}$ exists. Now $v, w \in \overline{a, b, d}$ and $p = \overline{a, b} \cap \overline{d, w}$ exist. Because $p \in \overline{a, b} = \overline{G}$ and $p, w \in \overline{d, c, y} \subset \overline{F}$ we have $p \in \overline{G} \cap \overline{F}$. □

LEMMA 3.3. (1) *For any four independent points $a, b, c, d \in M$ it holds that $\overline{a, b, c, d} = \bigcup_{x\in\overline{b,c,d}} \overline{a, x}$ and $\langle a, b, c, d\rangle = \bigcup_{x\in\overline{b,c,d}} \overline{a, x} \cap M$.*

(2) *Every 3-dimensional subspace T of M satisfies the exchange condition.*

Proof (1) Let $U := \bigcup_{x\in\overline{b,c,d}} \overline{a, x}$. Clearly $\overline{b, c, d} \subset \overline{a, b, c, d}$ and $\overline{a, x} \subset \overline{a, b, c, d}$ for every $x \in \overline{b, c, d}$, hence $U \subset \overline{a, b, c, d}$.

Clearly $a, b, c, d \in U$. We proof that U is a subspace of P. Then it follows that $\overline{a, b, c, d} \subset \overline{U} = U$. For distinct points $x, y \in \overline{b, c, d}$ let $a_x \in \overline{a, x}$, $a_y \in \overline{a, y}$ with $a \neq a_x, a_y$. Hence $E := \overline{a, x, y} \cap M$ and $F := \overline{b, c, d} \cap M$ are distinct planes of M with $x, y \in \overline{E} \cap \overline{F}$. By Lemma 3.2 for every $q \in \overline{a_x, a_y}$ it follows that $p = \overline{a, q} \cap \overline{F}$ exists. Hence $q \in \overline{a, p} \subset U$, $\overline{a_x, a_y} \subset U$ and U is a subspace. Since M is a locally complete restriction of P, $\overline{a, b, c, d} \cap M = \overline{\langle a, b, c, d\rangle} \cap M = \langle a, b, c, d\rangle = U \cap M = \bigcup_{x\in\overline{b,c,d}} \overline{a, x} \cap M$ by 2.1(2).

(2) There are $a, b, c, d \in T$ with $T = \langle a, b, c, d\rangle$. Let $a' \in \langle a, b, c, d\rangle \setminus \langle b, c, d\rangle$ be any point with $a \neq a'$. By (1) there exists an $x \in \langle b, c, d\rangle$ with $a' \in \overline{a, x}$, hence $a \in (\overline{a', x} \cap M) \subset \langle a', b, c, d\rangle$ by (1). □

THEOREM 3.4. *Let M be a locally complete restriction of $(P, \mathfrak{L})$ satisfying* (P4). *Let E, F be any planes of M with* $\dim\langle E \cup F\rangle = 3$ *and $E \cap F \neq \emptyset$. Then $E \cap F \in \mathfrak{L}(M)$ is a line.*

Proof Let $b \in E \cap F$, $c, d \in E$ with $E = \langle b, c, d\rangle$, hence $\overline{E} = \overline{b, c, d}$. Let $a \in F \setminus E$, i.e., a, b, c, d are independent and by Lemma 3.3(2), $\langle E \cup F\rangle = \langle a, b, c, d\rangle$. By Lemma 3.3(1), $F \subset \bigcup_{x\in\overline{E}} \overline{a, x} \cap M$. Hence for a point $z \in F \setminus \langle a, b\rangle$, i.e. $F = \langle a, b, z\rangle$, there is a point $x \in \overline{E}$ with $z \in \overline{a, x} \cap M$. By (G), there exists $p \in \overline{x, b} \cap M$ with $p \neq b$. We have $p \in E = \overline{E} \cap M$ and $p \in F = \langle a, b, z\rangle =$

$\overline{a, b, z} \cap M$. Therefore $\langle b, p\rangle \subset F \cap E$. Since $\dim\langle E \cup F\rangle = 3$, we have $E \neq F$, hence $\langle b, p\rangle = F \cap E$. □

THEOREM 3.5 *Let M be a locally complete restriction of* $(P, \mathfrak{L})$. *If* $\dim M \geqslant 3$, *then* (P2) *implies* (P1).

Proof. Since (P2) implies (P4), we can use Theorem 3.4 and Lemma 3.2. Let $E \subset M$ be a plane and $L_1, L_2 \subset \overline{E}$ be distinct lines. Since $\dim M \geqslant 3$ there is a point $a \in M \setminus E$. By (E), $F_i = \overline{L_i \cup \{a\}} \cap M$ are planes of M for $i = 1, 2$. By Theorem 3.4, the planes $F_1, F_2 \subset \langle E \cup \{a\}\rangle$ intersect in a line $G = F_1 \cap F_2$. By (P2), $x_i = \overline{G} \cap \overline{L_i}$ exist for $i = 1, 2$. Since $a \neq \overline{E}$ we have $|\overline{G} \cap \overline{E}| = 1$, hence $x_1 = x_2 \in L_1 \cap L_2$. □

THEOREM 3.6. *Let M be a locally complete restriction of* $(P, \mathfrak{L})$. *If* $\dim M \geqslant 3$ *and if*

(∗) *for any distinct planes $E, F \subset M$ containing distinct points $x, y \in \overline{E}, \overline{F}$ it holds that $\overline{E} \cap \overline{F} = \overline{x, y}$,*

is satisfied, then (P4) implies (P1).

Proof. Let $E \subset M$ be a plane and $L_1, L_2 \subset \overline{E}$ be distinct lines. Since $\dim M \geqslant 3$ there is a point $a \in M \setminus E$. By (E), $F_i = \overline{L_i \cup \{a\}} \cap M$ are planes of M for $i = 1, 2$ which by Theorem 3.4 intersect in a line $G = F_1 \cap F_2$. By Lemma 3.2 $x = \overline{G} \cap \overline{E}$ exists and by (∗) it follows $x \in \overline{F}_i \cap \overline{E} = L_i$ for $i = 1, 2$. Hence $x \in L_1 \cap L_2$. □

COROLLARY 3.7. *Let M be a locally complete restriction of* $(P, \mathfrak{L})$. *If* $\dim M \geqslant 3$, *then* (P3) *implies* (P1).

Proof If $\overline{E}$ satisfies the exchange condition for every plane $E \subset M$, obviously the property (∗) of Theorem 3.6 holds. □

We summarize 3.5 and 3.7:

THEOREM 3.8. *Let* $(M, \mathfrak{L}(M))$ *be a locally complete restriction of* $(P, \mathfrak{L})$ of $\dim M \geqslant 3$. *Then the properties* (P1), (P2) and (P3) *are pairwise equivalent*.

4. Characterization of a Locally Complete Restriction

In this section we consider properties which imply that $(P, \mathfrak{L})$ is a generalized projective space.

THEOREM 4.1. *Let* $(M, \mathfrak{L}(M))$ *be a locally complete restriction of* $(P, \mathfrak{L})$ *which satisfies* (P1), *or* (P2), *or* (P3), *respectively. If* $\dim P \geqslant 3$, *then* $(P, \mathfrak{L})$ *is a generalized projective space*.

Proof. By Theorem 2.4, dim $P \geqslant 3$ implies dim $M \geqslant 3$, hence by Theorem 3.8, (P1), (P2) and (P3) are equivalent.

We show the Veblen–Young–Axiom for P. Let $x_1, y_1, x_2, y_2, z \in P$ be distinct points with x_1, y_1, z collinear x_2, y_2, z collinear and $\overline{x_1, y_1} \neq \overline{x_2, y_2}$. We have to show that $\overline{x_1, x_2} \cap \overline{y_1, y_2} \neq \emptyset$.

Since dim $P \geqslant 3$ and $\overline{M} = P$ by Lemma 2.3, there exists a point $a \in M \setminus \overline{x_1, x_2, z}$. Let $b_i \in \overline{a, x_i} \cap M \setminus \{a\}$, $c_i \in \overline{a, y_i} \cap M \setminus \{a\}$ for $i = 1, 2$ and $d \in \overline{a, z} \cap M \setminus \{a\}$ (cf. (G)), then $c_1, c_2 \in \langle a, b_1, b_2, d\rangle = \overline{a, b_1, b_2, d} \cap M = \overline{a, x_1, x_2, z} \cap M$.

Since dim$\langle a, b_1, b_2, d\rangle = 3$, by Theorem 3.4 there is a point $e \in \langle a, b_1, b_2, d\rangle \setminus \{a\}$ with $\langle a, b_1, b_2\rangle \cap \langle a, c_1, c_2\rangle = \langle a, e\rangle$. Hence $\overline{a, e} \subset \overline{a, b_1, b_2} = \overline{a, x_1, x_2}$ and $\overline{a, e} \subset \overline{a, c_1, c_2} = \overline{a, y_1, y_2}$. By (P2) the points $w_x = \overline{a, e} \cap \overline{x_1, x_2}$ and $w_y = \overline{a, e} \cap \overline{y_1, y_2}$ exist.

Since $a \notin \overline{x_1, x_2, z} = \overline{x_1, x_2, y_1, y_2}$ it follows that $|\overline{a, e} \cap \overline{x_1, x_2, y_1, y_2}| = 1$ and $w := w_x = w_y$. Hence $w \in \overline{x_1, x_2}, \overline{y_1, y_2}$, this is, $\overline{x_1, x_2} \cap \overline{y_1, y_2} = w$. □

The existence of a point $a \notin \overline{x_1, x_2, z}$ is necessary for this proof. Therefore we must know that dim $P \geqslant 3$. Since by Theorem 2.4 we have dim $P \geqslant$ dim $M - 1$, we obtain:

COROLLARY 4.2. *Let* $(M, \mathfrak{L}(M))$ *be a locally complete restriction of* $(P, \mathfrak{L})$ *satisfying* (P1), *or* (P2), *or* (P3), *respectively. If* dim $M \geqslant 4$, *then* $(P, \mathfrak{L})$ *is a projective space.*

THEOREM 4.3. *Let* $(M, \mathfrak{L}(M))$ *be a locally complete restriction of* $(P, \mathfrak{L})$ of dim $M \geqslant 3$ *which satisfies* (P1), *or* (P2), *or* (P3), *respectively. Then the following statements are equivalent:*

(1) $(P, \mathfrak{L})$ *is a generalized projective space.*
(2) $(P, \mathfrak{L})$ *satisfies the exchange condition.*
(3) *The planes of* $(P, \mathfrak{L})$ *satisfy the exchange condition.*
(4) dim $P =$ dim M.
(5) dim $P \geqslant 3$.

Proof. As is known, a generalized projective space satisfies the exchange condition, hence it suffices to show (i) ‘2 ⇒ 4 ⇒ 5 ⇒ 1’ and (ii) ‘3 ⇒ 5’.

(i) By 2.5, if $(P, \mathfrak{L})$ satisfies the exchange condition, then dim $P =$ dim M, hence dim $P \geqslant 3$. By Theorem 4.1, dim $P \geqslant 3$ implies that $(P, \mathfrak{L})$ is a generalized projective space.
(ii) For a plane E of M we have by (E), $\overline{E} \cap M = E$. Therefore dim $M \geqslant 3$ implies $E \neq M$ and $\overline{E} \neq P$. Hence, if the planes of P satisfy the exchange condition, we have dim $P \geqslant 3$. □

COROLLARY 4.4. *Let $(M, \mathfrak{L}(M))$ be a locally complete restriction of a generalized projective space $(P, \mathfrak{L})$. If for every plane E of M, the plane $\overline{E}$ is a projective plane, then $(P, \mathfrak{L})$ is a projective space.*

Proof. Let $L \in \mathfrak{L}$ and $a \in M \setminus L$. Then $\overline{L \cup \{a\}}$ is a projective plane, hence $|L| \geqslant 3$. □

The question if $(P, \mathfrak{L})$ also for dim $M = 3$ satisfies the exchange condition, if dim $M =$ dim P or if $(P, \mathfrak{L})$ is a generalized projective space, respectively, is answered in the next section.

5. Projective Embedding

To handle the case dim $M = 3$, we will use Kahn's Theorem which state that every locally projective linear space $(M, \mathfrak{M})$ of dim $M = 3$ satisfying the Bundle Theorem is embeddable in a projective space. We recall that a subset $\mathfrak{b} \subset \mathfrak{L}(M)$ is called a bundle if any two lines $L, G \in \mathfrak{b}$ are coplanar in $(M, \mathfrak{L}(M))$ and if for every point $a \in M$ there is a line $L \in \mathfrak{b}$ with $a \in L$.

LEMMA 5.1. *Let $(M, \mathfrak{L}(M))$ be a locally complete restriction of $(P, \mathfrak{L})$ of* dim $M \geqslant 3$ *which satisfies* (P4). *Then:*

(1) *Let $A, B, L \in \mathfrak{L}(M)$ be pairwise coplanar lines not in a common plane, then $\overline{A} \cap \overline{B} = \overline{A} \cap \overline{L} = \overline{B} \cap \overline{L} \neq \emptyset$.*
(2) *For every $y \in P$, $[y] := \{\overline{y, a} \cap M : a \in M\}$ is a bundle of $(M, \mathfrak{L}(M))$.*
(3) *Let $\mathfrak{b} \subset \mathfrak{L}(M)$ be a bundle, then there is a point $x \in P$ with $\mathfrak{b} = \{\overline{x, a} \cap M : a \in M \setminus \{x\}\} = [x]$.*

Proof. (1) By (P4), $x = \overline{A} \cap \overline{B}$ exists and $\overline{A} \cap \overline{L} \neq \emptyset \neq \overline{B} \cap \overline{L}$. Since A, B, L are not coplanar, by (E) also $\overline{L} \not\subset \overline{A \cup B}$, hence $|\overline{L} \cap \overline{A \cup B}| \leqslant 1$. It follows that $x = \overline{A} \cap \overline{L} = \overline{B} \cap \overline{L}$

(2) For $a, b \in M \setminus \{y\}$, the lines $\overline{a, y}, \overline{b, y}$ are coplanar, by (G) $\overline{a, y} \cap M, \overline{b, y} \cap M \in \mathfrak{L}(M)$, and by (E), $\overline{a, y} \cap M, \overline{b, y} \cap M$ are coplanar in M.

(3) Let $a, b \in M$ with $[a], [b] \neq \mathfrak{b}$, and let $A, B \in \mathfrak{b}$ with $a \in A, b \in B$. Since dim $M \geqslant 3$ there is a point $c \in M \setminus \langle A \cup B \rangle$ and a line $C \in \mathfrak{b}$ with $c \in C$. The lines A, B, C are not coplanar. Hence by (1), $x = \overline{A} \cap \overline{B} = \overline{A} \cap \overline{C} = \overline{B} \cap \overline{C}$ exists. For every line $G \in \mathfrak{b}$, G is not coplanar with A, B, or A, C, or B, C, respectively. Therefore by (1), $x \in \overline{G}$. Hence by (G), $\mathfrak{b} = [x]$.

THEOREM 5.2. *Let $(M, \mathfrak{L}(M))$ be a locally complete restriction of $(P, \mathfrak{L})$ of* dim $M \geqslant 3$ *which satisfies* (P4). *Then:*

(1) *$(M, \mathfrak{L}(M))$ is locally generalized projective.*
(2) *$(M, \mathfrak{L}(M))$ satisfies the Bundle Theorem.*

Proof. (1) Let $x \in M$. For a subspace E of M we denote by $\widehat{E} := \big\{\langle a, x\rangle : a \in E \setminus \{x\}\big\}$ a subset of $[x]$. Let $\mathfrak{L}_x := \{\widehat{E} : E$ a plane of M with $x \in E\}$. Since the planes of $(M, \mathfrak{L}(M))$ satisfy by 3.1(2) the exchange condition, $([x], \mathfrak{L}_x)$ is a linear space. For a subspace $T \subset M$ of $\dim T = 3$ containing x let $a, b, c \in T$ with $T = \langle a, b, c, x\rangle$ (cf. 3.3 (2)). Clearly $\widehat{T}$ is a subspace of $([x], \mathfrak{L}_x)$ which is generated by $\langle a, x\rangle$, $\langle b, x\rangle$, $\langle c, x\rangle$, since for any two lines of T the plane generated by the lines is contained in T. Therefore $\widehat{T}$ is a plane of $([x], \mathfrak{L}_x)$. By Theorem 3.4, $\widehat{T}$ is a generalized projective plane.

(2) Let $A, B, C, D \in \mathfrak{L}(M)$, no three in a common plane, and let $\{A, B\}$, $\{A, C\}$, $\{A, D\}$, $\{B, C\}$, and $\{B, D\}$ be pairwise coplanar. By 5.1(1), $x = \overline{A} \cup \overline{B}$ exists with $x \in \overline{C}$ and $x \in \overline{D}$, hence $x = \overline{C} \cap \overline{D}$. By (E), C, D are coplanar. □

We define $P' := \{\mathfrak{b} : \mathfrak{b}$ is a bundle of $(M, \mathfrak{L}(M))\}$. For any two bundles $\mathfrak{x}, \mathfrak{y} \in P'$ let $[\mathfrak{x}, \mathfrak{y}] := \{\mathfrak{z} \in P' :$ for every $a \in M$ with $[a] \neq \mathfrak{x}, \mathfrak{y}, \mathfrak{z}$ the lines of $\mathfrak{x}, \mathfrak{y}, \mathfrak{z}$ through a are coplanar in $(M, \mathfrak{L}(M))\}$.

We define $\mathfrak{L}' := \{[\mathfrak{x}, \mathfrak{y}] : \mathfrak{x}, \mathfrak{y} \in P'$ distinct $\}$.

By the Theorem of Kahn and by 1.1 we know that $(P', \mathfrak{L}')$ is a generalized projective space, if $(M, \mathfrak{L}(M))$ is locally generalized projective, if $(M, \mathfrak{L}(M))$ satisfies the Bundle Theorem, and if $\dim M \geqslant 3$ (cf. [2, 5]).

LEMMA 5.3. *Let $(M, \mathfrak{L}(M))$ be a locally complete restriction of $(P, \mathfrak{L})$ of* $\dim M \geqslant 3$ *which satisfies* (P1), *or* (P2), *or* (P3), *respectively. Then three points $x, y, z \in P$ are collinear in P if and only if for every $a \in M \setminus \{x, y, z\}$ the lines $\overline{a, x} \cap M$, $\overline{a, y} \cap M$, $\overline{a, z} \cap M$ are coplanar in $(M, \mathfrak{L}(M))$.*

Proof. If x, y, z are collinear and $x \neq y$, then $z \in \overline{a, x, y}$ and by (E) the lines $\overline{a, x} \cap M$, $\overline{a, y} \cap M$, $\overline{a, z} \cap M$ are coplanar.

Assume $z \notin \overline{x, y}$. Let $a \in M \setminus \overline{x, y}$, then $E := \overline{a, x, y} \cap M$ is a plane of M. Since $\dim M \geqslant 3$, there exists $b \in M \setminus E = M \setminus \overline{E}$ and $F := \overline{b, x, y} \cap M$ is a plane of M. By Theorem 3.8, (P1) is satisfied and $\overline{E}, \overline{F}$ are projective planes. Since $b \notin \overline{E}$, $\overline{E} \cap \overline{F} = \overline{x, y}$. Hence $z \notin \overline{x, y}$ implies $z \notin \overline{E}$ or $z \notin \overline{F}$. Therefore $\overline{a, z} \not\subset E$ or $\overline{b, z} \not\subset F$, and $\overline{a, x} \cap M$, $\overline{a, y} \cap M$, $\overline{a, z} \cap M$ are not coplanar, or $\overline{b, x} \cap M$, $\overline{b, y} \cap M$, $\overline{b, z} \cap M$ are not coplanar. □

THEOREM 5.4. *Let $(M, \mathfrak{L}(M))$ be a locally complete restriction of $(P, \mathfrak{L})$ of* $\dim M \geqslant 3$ *which satisfies* (P1), *or* (P2), *or* (P3), *respectively, and let $(P', \mathfrak{L}')$ denote the bundle space. Then $\phi \colon P \to P'$, $y \mapsto [y]$ is an isomorphism.*

Proof. By 5.1(2), for every $y \in P$, $[y]$ is a bundle. Clearly $[y] \neq [z]$ for $y \neq z$. By 5.1(3), for every bundle $\mathfrak{b}$ there exists a point $x \in P$ with $\mathfrak{b} = [x]$, hence $\phi \colon P \to P'$, $y \mapsto [y]$ is a bijection. By 5.3 three points $x, y, z \in P$ are collinear if and only if for every $a \in M \setminus \{x, y, z\}$ the lines $\overline{a, x} \cap M$, $\overline{a, y} \cap M$, $\overline{a, z} \cap M$ are coplanar in $(M, \mathfrak{L}(M))$. This is equivalent to the collinearity of the bundles $[x], [y], [z]$ in $(P', \mathfrak{L}')$ by definition and by 5.1(2). □

THEOREM 5.5. *Let* $(M, \mathfrak{L}(M))$ *be a locally complete restriction of* $(P, \mathfrak{L})$ *of* dim $M \geqslant 3$ *satisfying* (P1), *or* (P2), *or* (P3), *respectively. Then* $(P, \mathfrak{L})$ *is a generalized projective space.*

Proof. We know by Theorem 5.2 that $(M, \mathfrak{L}(M))$ is locally generalized projective and satisfies the Bundle Theorem. Hence the bundle space $(P', \mathfrak{L}')$ of M is a generalized projective space by [2, 5] and Lemma 1.1. By Lemma 5.4, $(P', \mathfrak{L}')$ and $(P, \mathfrak{L})$ are isomorphic. □

COROLLARY 5.6. *Let* $(M, \mathfrak{L}(M))$ *be a locally complete restriction of* $(P, \mathfrak{L})$. *If for every plane* E *of* M *the plane* $\overline{E}$ *of* P *generated by* E *is a projective plane, then* $(P, \mathfrak{L})$ *is a projective space.*

Proof. For dim $M = 2$, $\overline{M} = P$ is a projective plane. For dim $M \geqslant 3$, $(P, \mathfrak{L})$ is a projective space by Theorem 5.5 and Corollary 4.4.

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