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16-dimensional compact projective planes with a large group fixing two points and two lines

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Abstract. We determine all planes having the properties of the title with a group of dimension at least 33.

Let $\mathcal{P} = (P, \mathfrak{L})$ be a topological projective plane with a compact point set P of finite (covering) dimension $d = \dim P > 0$. A systematic treatment of such planes can be found in the book *Compact Projective Planes* [21]. Each line $L \in \mathfrak{L}$ is homotopy equivalent to a sphere $\mathbb{S}_\ell$ with $\ell \mid 8$, and $d = 2\ell$, see [21, (54.11)]. In all known examples, L is in fact homeomorphic ($\approx$) to $\mathbb{S}_\ell$. Taken with the compact-open topology, the automorphism group $\Sigma = \mathrm{Aut}\,\mathcal{P}$ (of all continuous collineations) is a locally compact transformation group of P with a countable basis [21, (44.3)]. The covering dimension $\dim \Sigma$ is an important parameter for characterizations of such planes. (For readers which are more familiar with the inductive dimension, we remark that for a locally compact group Λ, the inductive dimension $\mathrm{ind}\,\Lambda$ coincides with $\dim \Lambda$ and with the dimension of the connected component Λ^1, cf. [21, (93.5, 6)].

The classical examples are the planes $\mathcal{P}_\mathbb{K}$ over the 3 locally compact, connected fields $\mathbb{K}$ with $\ell = \dim \mathbb{K}$ and the 16-dimensional Moufang plane $\mathcal{O} = \mathcal{P}_\mathbb{O}$ over the octonion algebra $\mathbb{O}$. If $\mathcal{P}$ is a classical plane, then $\mathrm{Aut}\,\mathcal{P}$ is an almost simple Lie group of dimension C_ℓ, where $C_1 = 8$, $C_2 = 16$, $C_4 = 35$, and $C_8 = 78$.

In all other cases, $\dim \Sigma \leqq \frac{1}{2}C_\ell + 1 \leqq 5\ell$. Planes with small groups abound, those with a group of dimension sufficiently close to $\frac{1}{2}C_\ell$ can be described explicitly. More precisely, the classification program seeks to determine all pairs $(\mathcal{P}, \Delta)$, where Δ is a connected closed subgroup of $\mathrm{Aut}\,\mathcal{P}$ and $b_\ell \leqq \dim \Delta \leqq 5\ell$ for a suitable bound $b_\ell \geqq 4\ell - 1$. This has been accomplished for $\ell \leqq 2$ and also for $b_4 = 17$. Results in the case $\ell = 8$ are as yet less satisfactory, and it is this case that will be considered here.

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Most theorems that have been obtained so far require additional assumptions on the structure and/or the action of Δ. If $\dim \Delta \geqq 27$, and in the relevant dimension range in particular, Δ is always a Lie group [15]. By the structure theory of Lie groups, Δ is semi-simple, or Δ contains a central torus subgroup, or Δ has a minimal normal vector subgroup, cf. [21, (94.26)]. The first two cases are understood fairly well:

(a) *If* Δ *is semi-simple and* $\dim \Delta > 28$*, then there are the following cases:* $\mathcal{P} \cong \mathcal{O}$*, or* $\Delta \cong \mathrm{SL}_3\mathbb{H}$ *has no fixed point and no fixed line and* $\mathcal{P}$ *is a Hughes plane* (as described in [21, § 86]), *or* $\Delta \cong \mathrm{Spin}_9(\mathbb{R}, r)$ *with* $r \leqq 1$ *contains a central reflection, whose center and axis are the only fixed point and the only fixed line of* Δ*, see* [13], [14].

(b) *If* Δ *contains a central torus, and if* $\dim \Delta > 30$*, then the commutator subgroup* Δ' *is isomorphic to* $\mathrm{SL}_3\mathbb{H}$ *and* (a) *applies to* Δ', see [17].

In the third case, Δ fixes a point or a line, cf. [5, (XI.10.19)] together with [21, (83.4, 6)]. Hence (a) and (b) imply

(c) *If* $\dim \Delta > 30$ *and* Δ *has no fixed element, then* $\mathcal{P}$ *is a Hughes plane or* $\mathcal{P} \cong \mathcal{O}$.

The case that Δ fixes exactly one element has been treated in [19]:

(d) *If* $\dim \Delta \geqq 35$ *and if* Δ *fixes one line and no point, then* $\mathcal{P}$ *is a translation plane*. All these translation planes have been determined in [6], [7], [9]. The classification shows that either $\mathcal{P} \cong \mathcal{O}$ or $\dim \Delta = 35$.

Little progress has been made in the cases where Δ fixes exactly two elements, necessarily a point and a line. If $\dim \Delta \geqq 40$, then $\mathcal{P}$ and its dual are translation planes [21, (87.7)], and all translation planes with $\dim \Delta \geqq 38$ are described in [21, (82.28)].

From now on, assume that $33 \leqq \dim \Delta < 40$ and that Δ fixes 2 points.

The stiffness theorems in [21, (83.23 or 26)] show that the stabilizer $\triangledown$ of a triangle satisfies $\dim \triangledown \leqq 30$. Therefore, under the above assumption, either Δ fixes exactly one line (which then contains all fixed points), or Δ fixes exactly two points and two lines. The present paper deals with the latter case. The case of only one fixed line and exactly two fixed points will be discussed elsewhere. The case of more than two fixed points is already settled:

(e) *If* Δ *fixes* (*at least*) 3 *points, then* $\dim \Delta \geqq 37$ *and* $\mathcal{P} \cong \mathcal{O}$, see [20].

In the octonion plane, a semi-simple subgroup of Σ fixing two points has dimension at most 28, see [21, (12.17, 17.13)]. Therefore, under the general assumption above, the next result follows from [18] together with (a). It is crucial for the subsequent arguments.

Lemma. *Up to duality,* Δ *has a minimal normal subgroup* $\Theta \cong \mathbb{R}^t$ *consisting of axial collineations with a common axis* W*, and* Θ *is a group of translations, or* $t = 1$.

Bödi's improvement [1] of [21, (83.23)] is particularly important:

($\square$) *If the fixed elements of the connected Lie group Λ form a connected subplane $\mathcal{E}$, then Λ is isomorphic to the* 14*-dimensional compact group* G_2 *or its subgroup* $\mathrm{SU}_3\mathbb{C}$ *or* $\dim \Lambda < 8$. *If $\mathcal{E}$ is a Baer subplane* ($\dim \mathcal{E} = 8$), *then Λ is a subgroup of* $\mathrm{SU}_2\mathbb{C}$. *Moreover,* $\Lambda \cong \mathrm{G}_2$ *implies* $\dim \mathcal{E} = 2$.

The following shall be proved here:

Theorem 1. *If* $\dim \Delta \geqq 33$ *and if Δ fixes exactly* 2 *points u, v and* 2 *lines $W = uv$ and $Y = av$, where the points a, u, v form a triangle, then the translation group $\mathsf{T} = \Delta_{[v,W]}$ is transitive, the complement Δ_a of T has a compact commutator group* $\Phi \cong \mathrm{Spin}_8\mathbb{R}$, *and* $\dim \Delta \geqq 36$. *If even* $\dim \Delta \geqq 38$, *then* $\mathcal{P} \cong \mathcal{O}$.

Corollary. *If a non-classical plane $\mathcal{P}$ has a group Δ as in Theorem* 1*, then the full automorphism group Σ of $\mathcal{P}$ has dimension at most* 37.

All planes satisfying the conditions of this theorem will be described in Theorem 2.

It seems quite unlikely that the classification problem in the case of 2 fixed points and 2 fixed lines has a reasonable solution for a lower bound of $\dim \Delta$ than 33. In fact, the assumption $\dim \Delta \geqq 33$ is needed in the lemma and it is used repeatedly in the proof of Theorem 1 in an essential way. Moreover, for other fixed configurations, even larger bounds are required for obtaining classification results.

The dimension formula [21, (96.10)] has a useful corollary for solvable groups:

Remark. Assume that Γ is a solvable Lie subgroup of Δ. Then Γ has a chain of normal subgroups Γ_κ with $\dim \Gamma_{\kappa+1}/\Gamma_\kappa \leqq 2$, see [2, I § 5, Th. 1, Cor. 4, p. 46]. *If κ is the largest index such that $a^{\Gamma_\kappa} = a$, if $\mathsf{N} = \Gamma_{\kappa+1}$ and $a \neq x \in a^{\mathsf{N}}$, then* $\dim x^{\Gamma_a} \leqq 2$. In fact, $x^{\Gamma_a} \subseteqq a^{\mathsf{N}}$ and $\dim x^{\Gamma_a} \leqq \dim \mathsf{N}/\mathsf{N}_a \leqq \dim \mathsf{N}/\Gamma_\kappa$.

Another fact that will be needed repeatedly is the

Observation. *If a maximal semi-simple subgroup Ψ of Δ (a Levi complement of the radical $\sqrt{\Delta}$) has a subgroup* $\Lambda \cong \mathrm{G}_2$, *then Ψ is almost simple, and $\Psi = \Lambda$ or there is a group* $\Upsilon \cong \mathrm{Spin}_7\mathbb{R}$ *with $\Lambda < \Upsilon \leqq \Psi$. The central involution $\alpha \in \Upsilon$ is a reflection.*

Proof. If Ψ is not almost simple, then some factor of Ψ contains a subgroup $\Gamma \cong \Lambda$. Since $W^\Gamma = W$, the involutions in Γ are planar, or else Γ would contain a reflection with axis W and Γ would not be simple. According to [21, (55.6)], the line W is an 8-sphere, and then the fixed elements of Γ form a 2-dimensional subplane $\mathcal{E}$ by [21, (96.35)]. There is an almost simple factor of Ψ which centralizes Γ and acts non-trivially on $\mathcal{E}$, but such a group never fixes two distinct points of $\mathcal{E}$, see [21, (38.3)]. Hence Ψ is almost simple. Inspection of the lists [21, (94.33 and 95.10)] of almost simple groups and their representations shows that the only possibilities in dimension at most 39 are groups of type B or D or the complexification of G_2. By [21, (96.35) and (94.34)], the latter group would fix all elements of the plane $\mathcal{E}$, but this contradicts (□). Because the group $SO_5\mathbb{R}$ does not act non-trivially on any compact projective plane [21, (55.40)], we conclude that $\Psi \cong \mathrm{Spin}_n(\mathbb{R}, r)$ with $0 \leqq r \leqq n-7$. The central involution $\alpha \in \Upsilon$ is not planar, or else Υ would induce a group $SO_7\mathbb{R}$ on the fixed plane of α. By [21, (55.29)] it follows that α is a reflection. □

N o t a t i o n. If the point set S contains a quadrangle, then $\langle S \rangle$ denotes the smallest *closed* subplane of $\mathcal{P}$ containing S. As customary, $\mathcal{P}^W$ means the affine plane obtained from $\mathcal{P}$ by removing the line W, see [21, (21.9)]. Let $L_2 = \{(t \mapsto at+b) : \mathbb{R} \to \mathbb{R} \mid a > 0,\ b \in \mathbb{R}\}$ be the unique non-commutative, 2-dimensional connected Lie group. The commutator subgroup Γ' of the group Γ should not be confused with the connected component Γ^1.

P r o o f o f T h e o r e m 1. According to the Lemma, Δ has a minimal normal subgroup $\Theta \cong \mathbb{R}^t$ of axial collineations with a common axis, and this axis is a fixed line of Δ.

1) Assume first that the elements of Θ have axis $Y = av$. Then $\Theta \leqq \Delta_{[u,Y]}$ is a group of homologies, and $t = 1$ by [21, (61.2)]. The connected component of Δ_a will be denoted by $\triangledown$. Let $c \in Y \setminus \{a, v\}$ and $w \in W \setminus \{u, v\}$.

2) If Λ is the connected component of the stabilizer $\triangledown_{c,w}$, then $\Gamma = \Lambda \cap \mathrm{Cs}\,\Theta$ fixes the orbit w^Θ pointwise. Hence the fixed elements of Γ form a connected subplane $\mathcal{F}$ and (□) applies. The dimension formula gives $\dim \Lambda \geqq 33 - 3{\cdot}8$, and $\Lambda/\Gamma \leqq \mathrm{Aut}\,\Theta$ implies that $\dim \Gamma \geqq \dim \Lambda - 1 \geqq 8$. Therefore, $\mathcal{F}$ is at most 4-dimensional by (□). The fixed plane $\mathcal{E}$ of Λ is a closed subplane of $\mathcal{F}$, and $\mathcal{E}$ is connected by [21, (55.4a)]. Now the stiffness theorem (□) shows that $\Lambda \cong G_2$, and this is true for each admissible choice of c and w.

3) Consider the connected component Ω of Δ_w and assume that Λ is a Levi complement of the radical $\sqrt{\Omega}$. Then $\dim \sqrt{\Omega} \geqq 11$, and the Remark shows that $\dim (\Lambda \cap \sqrt{\Omega}) > 0$ for a suitable choice of c, but Λ is a simple group. Hence, according to the Observation, Ω has a subgroup $\Upsilon \cong \mathrm{Spin}_7\mathbb{R}$. Because Ω fixes 3 points on W, the reflection $\alpha \in \Upsilon$ has axis W and some center $x \in Y$. Since $x^\Delta \neq x$, the conjugacy class α^Δ has positive dimension, and, by a well-known fact, $\alpha^\Delta \alpha$ is contained in the translation group $\mathsf{T} = \Delta_{[v,W]}$.

4) The group T might be non-abelian. By [21, (55.28)], each finite subgroup of T is trivial. In particular, T has no torus subgroup, and the connected component T^1 of T is a simply connected Lie group, cf. [21, (93.10 and 94.31)]. If T^1 has a radical $\mathsf{P} \neq \mathbb{1}$, then one term of the derived series of P is a normal vector subgroup of Δ. If T^1 is semi-simple, however, then T^1 has an almost simple factor X which is isomorphic to the simply connected covering group of $SL_2\mathbb{R}$, see [21, (94. 28 and 37)]. In this case, the infinite center Z of X is contained in the center of Δ, and Ω_a fixes each point of the orbit a^{Z}. In particular, Ω_a fixes a quadrangle. Since $\dim \Omega_a \geqq 33-2{\cdot}8$, this contradicts the stiffness theorem [21, (83.23)].

5) Consequently, there exists always a minimal normal vector subgroup $\Theta \leqq \mathsf{T} = \Delta_{[v,W]}$. If $\mathbb{R} \cong \Pi \leqq \Theta$ and $\mathbb{1} \neq \varrho \in \Pi$, and if $c = a^{\varrho} \in Y$, then the connected component Λ of $\Omega_{a,c}$ centralizes Π and hence fixes a connected subplane $\mathcal{E}$ pointwise, and $\dim \Lambda \geqq 17 - t > 8$. The stiffness theorem ($\square$) gives $\Lambda \cong \mathrm{G}_2$ and $t \geqq 3$. The group $\Theta \cap \mathrm{Cs}\,\Lambda$ contains Π and acts as a one-dimensional translation group on the 2-dimensional plane $\mathcal{E}$. Therefore, Λ has a non-trivial representation on Θ and $\dim \Theta > 7$, see [21, (95.10)]. It follows that $\Theta = \mathsf{T} \cong \mathbb{R}^8$.

6) The group Π can be chosen arbitrarily. Hence Ω_a is an irreducible subgroup of $\mathrm{Aut}\,\mathsf{T}$. By [21, (95.6b)], the commutator group Ω_a' is semi-simple, and $16 \leqq \dim \Omega_a' \leqq 22$. From [21, (95.10)] or the Observation it follows that $\Omega_a' \cong \mathrm{Spin}_7\mathbb{R}$, in particular, $\dim \Omega_a \geqq 21$.

7) Consider the stabilizer $\triangledown = \Delta_a$ of the triangle a, u, v and remember that $\dim \triangledown \leqq 30$. By [21, (96.35)], there is an orbit $z^{\Lambda} \approx \mathbb{S}_6$ in W. Interchanging the rôles of z and w shows $\dim w^{\triangledown} \geqq 6$. Since $\dim \triangledown_w = \dim \Omega_a$, step 6) and the dimension formula give $\dim \triangledown \geqq 27$. The kernel $\mathsf{K} = \triangledown \cap \mathrm{Cs}\,\mathsf{T}$ of the action of $\triangledown$ on T is a group of homologies with axis Y. The connected component of K is either compact or two-ended, and a two-ended group is isomorphic to $\mathbb{R} \times C$ where C is compact, see [21, (61.2)]. Therefore, K contains a compact, connected normal subgroup Ψ of $\triangledown$ such that $\dim(\mathsf{K}/\Psi) \leqq 1$. Moreover, $\dim \Psi < 8$, or Ψ would be transitive on $uv \setminus \{u, v\}$. Either the commutator $[\Lambda, \Psi] = \mathbb{1}$, or Λ acts faithfully on Ψ and induces an irreducible representation on the Lie algebra of Ψ. In the latter case, $\dim \Psi = 7$, and Ψ is not semi-simple. It follows that Ψ has a torus factor ([21, (94.31c)]) and then that Ψ is itself a torus, but this would imply $[\Lambda, \Psi] = \mathbb{1}$ after all, cf. [21, (93.19)]. We conclude that $[\mathsf{K}, \Lambda] = \mathbb{1}$, and K leaves the subplane $\mathcal{E}$ invariant. Consequently, $\dim \mathsf{K} \leqq 1$.

8) By [21, (95.6)], the commutator group $(\triangledown/\mathsf{K})'$ is a semi-simple subgroup of $\mathrm{Aut}\,\mathsf{T}$ which contains $\mathrm{Spin}_7\mathbb{R}$, and $\dim(\triangledown/\mathsf{K})' \geqq 25$. From the list [21, (95.10)], it follows that $(\triangledown/\mathsf{K})' \cong \mathrm{SO}_8(\mathbb{R}, r)$ with $r \leqq 1$, and then $\triangledown' \cong \mathrm{Spin}_8(\mathbb{R}, r)$. (Remember that Δ has no subgroup $\mathrm{SO}_5\mathbb{R}$, so that $\triangledown' \ncong \mathrm{SO}_8(\mathbb{R}, r)$).

9) We may assume that $\Delta = \triangledown'\mathsf{T}$, and then $\dim \Delta = 36$ and $\triangledown \cong \mathrm{Spin}_8(\mathbb{R}, r)$, $r \leqq 1$. The possibility $r = 1$ will lead to a contradiction. If $r = 1$, then $\triangledown$ induces on T and on the line av the orthogonal group $\mathrm{SO}_8(\mathbb{R}, 1)$, and $\triangledown$ leaves some cone in T invariant.

10) The group $\Gamma = (\triangledown_w)^1$ acts faithfully on T and $\Gamma' \cong \mathrm{Spin}_7\mathbb{R}$ by step 6). The stiffness results [5, XI.9.9] or ($\square$) imply $\Gamma_c^1 \cong \mathrm{G}_2$ for each $c \in av \setminus \{a, v\}$. It follows that Γ' has only 7-dimensional orbits on the sphere S consisting of the rays in the vector group T. Hence $\triangledown_w$ is transitive on S and there is no $\triangledown$- invariant cone.

11) Now let $\dim \Delta \geqq 38$. The stiffness theorem implies $\dim \triangledown = 30$ and $\dim \Delta = 38$. Moreover, for each point $x \notin D := au \cup av \cup uv$, the stabilizer $\triangledown_x$ fixes a quadrangle and satisfies $\dim \triangledown_x = 14$. Hence $\dim x^{\triangledown} = 16$ and $x^{\triangledown}$ is open in P by [21, (96.11)]. Consequently, $\triangledown$ is transitive outside of D, and Δ induces on $av \setminus \{v\}$ a doubly transitive group Δ/K. Because $\triangledown' \cong \mathrm{Spin}_8\mathbb{R}$, it follows from [21, (95.6 and 10) or (96.16)] or

from [23] that $\triangledown/\mathsf{K} \cong e^{\mathbb{R}} \times \mathrm{SO}_8\mathbb{R}$. The kernel of the action is $\mathsf{K} = \Delta_{[u,av]} \cong \mathbb{R}^\times$. In the next step it will follow that K is contained in the center of Δ.

12) Whenever $\mathcal{E}$ is the (2-dimensional) fixed plane of a subgroup $\Lambda \cong \mathrm{G}_2$ of $\triangledown$, the radical $\mathsf{P} = \sqrt{\triangledown}$ acts faithfully on $\mathcal{E}$, and $\mathsf{P} \cong \mathbb{R}^2$ by [21, (33.10)]. In fact, P is contained in the center of $\triangledown$ since $\triangledown = \mathsf{P} \times \triangledown'$. Consequently, the stabilizer P_w fixes each point of the orbit $w^{\triangledown} = W \setminus \{u, v\}$, and $\mathsf{P}_w \leqq \triangledown_{[W]}$. The action of P on $\mathcal{E}$ shows $\mathsf{P}_w \cong \mathbb{R}$. There is a unique reflection α with axis W in $\triangledown'$ and $\triangledown_{[W]} = \mathsf{P}_w \times \langle\alpha\rangle \cong \mathbb{R}^\times$. Dually, $\mathbb{R} \cong \mathsf{P}_b < \triangledown_{[v]} \cong \mathbb{R}^\times$ for $b \in au \setminus \{a, u\}$.

13) By steps 5) and 11), the group Δ is transitive on the set $\mathfrak{F}$ of all flags (p, H) with $p \notin W, Y$ and $u, v \notin H$. Letting $H = aw$, where $a \in Y$, we find that $\mathsf{H} := \Delta_H = \triangledown_w = \mathsf{P}_{[a]} \times \triangledown'_w$. Dually, $\Pi := \Delta_p = \triangledown_b^\xi$ for a suitable $\xi \in \mathsf{T}$. Note that $\mathsf{H}' \cong \Pi' \cong \mathrm{Spin}_7\mathbb{R}$ by step 3). The geometry which has $\mathfrak{F}$ as incidence relation can be reconstructed from the triple $(\Delta, \mathsf{H}, \Pi)$ by a procedure due to Freudenthal [3, § 6], cf. also [22]. This geometry is obtained from the projective plane $\mathcal{P}$ by deleting two lines and two pencils and thus uniquely determines $\mathcal{P}$.

14) Since Δ is isomorphic to the stabilizer of two points and two lines (with the same incidence relations as in the theorem) of the classical Moufang plane, it suffices to show that the automorphism group $\mathsf{A} = \mathrm{Aut}\,\Delta$ is transitive on the set of all admissible pairs (H, Π) in Δ in order to prove that $\mathcal{P} \cong \mathcal{O}$. This will be done in the next steps.

15) A subgroup $\Phi \cong \mathrm{Spin}_8\mathbb{R}$ of Δ has a center of order 4 containing 3 reflections. Therefore, Φ fixes a triangle, and we may assume that $\Phi = \triangledown'$. The action of Φ on the Moufang plane $\mathcal{O}$ is described in detail in [21, (17.16)]. In particular, Φ has 3 conjugacy classes of subgroups $\mathrm{Spin}_7\mathbb{R}$, and these are permuted cyclically by the *triality* automorphism. All groups in the conjugacy class of the group Υ from step 3) have the reflection α with center a and axis W in common. They induce the group $\mathrm{SO}_7\mathbb{R}$ on W and act faithfully on the other two sides of the fixed triangle D. By [21, (96.36)], the action of Υ on W is linear, and the fixed elements of Υ on W form a circle. Since $\mathrm{Aut}\,\mathcal{O}$ is transitive on quadrangles, see [21, (17.6)], it follows that A is transitive on the set of all pairs (H', Π') of commutator groups of admissible pairs (H, Π).

16) If (H, Π) is admissible, then $\mathsf{H} \cap \Pi = \Lambda \cong \mathrm{G}_2$, and the fixed elements of Λ form a 2-dimensional subplane $\mathcal{E}$. For a given intersection Λ, each of the 3 conjugacy classes mentioned in step 15) contains a unique group $\Upsilon \cong \mathrm{Spin}_7\mathbb{R}$ such that $\Lambda < \Upsilon < \Phi$.

17) Consider the one-parameter group $\Xi = \mathsf{T} \cap \mathrm{Cs}\,\Lambda$ and the 3-dimensional group $\Gamma = \Xi\,\mathsf{P}$. Because Λ contains commuting Baer involutions, it follows from [21, (83.10)] that the kernel of the action of Γ on $\mathcal{E}$ is compact and hence trivial, moreover, $\Xi = \Gamma'$ (since $\Xi\,\mathsf{P}_w \cong \mathrm{L}_2$ and $\Gamma/\Xi \cong \mathsf{P}$ is commutative). Planes $\mathcal{E}$ admitting such a group have been determined in [16], see also [21, § 37]. The existence of 3 reflections in the center of Φ implies that $\mathcal{E} \cong \mathcal{P}_{\mathbb{R}}$.

18) On the affine plane $\mathcal{E} \setminus W \cong \mathbb{R}^2$ the elements of Γ induce the maps

$$\gamma(r, s, t) : (x, y) \mapsto (e^r x, e^s y + t).$$

The center Z of Γ is characterized by $s=t=0$ and the commutator group Γ' by $r=s=0$. Any complement of $\Gamma'\mathsf{Z}$ in Γ is of the form $\mathsf{E}(d,q)=\{\gamma(ds,\ s,\ q(e^s-1)) \mid s\in\mathbb{R}\}$. One can easily verify that the following maps are automorphisms of Γ:

$$\gamma(r,s,t)\mapsto\gamma(fr+gs,\ s,\ ht+k(e^s-1)) \text{ where } f,g,h,k\in\mathbb{R} \text{ and } fh\neq 0.$$

19) If $p=(1,1)$ and $H=ap$, then $\mathsf{H}_{\mathcal{E}}=\mathsf{H}\cap\mathrm{Cs}\,\Lambda=\mathsf{E}(1,0)$ is a group of homologies with axis W and center $a=H\cap Y$, dually, $\Pi_{\mathcal{E}}=\Pi\cap\mathrm{Cs}\,\Lambda=\mathsf{E}(0,-1)$ consists of homologies with center v and axis pu. Such complements of $\Gamma'\mathsf{Z}$ are never conjugate; since $a\notin pu$, they do not centralize each other. Consequently, these properties express the fact that $(\mathsf{H}_{\mathcal{E}},\Pi_{\mathcal{E}})$ is an admissible pair. The automorphism group of Γ is transitive on the set of admissible pairs of complements of $\Gamma'\mathsf{Z}$. In fact, the automorphisms of Γ induce linear mappings of the parameters d and q which characterize these complements.

20) It remains to be shown that the automorphisms of Γ extend to automorphisms of Δ. This is obvious except for automorphisms of the form $\gamma(r,s,t)\mapsto\gamma(r,s,\ t\ +\ k(e^s-1))$. Remember that $\Delta=\bigtriangledown\mathsf{T}$ and that $\bigtriangledown'$ induces on $\mathsf{T}\cong\mathbb{R}^8$ the group $\mathrm{SO}_8\mathbb{R}$, and consider the inner automorphism $\widehat{\eta}$ of Δ induced by $\eta=\gamma(0,0,k)\in\mathsf{T}\cap\mathrm{Cs}\,\Lambda=\Xi$. Obviously, $\widehat{\eta}$ induces on $\Gamma=\Xi\,\mathsf{P}$ the automorphism in question. Since $\Lambda^\eta=\Lambda$, it follows from 16) that $\widehat{\eta}$ fixes the pair (H',Π'). Together with step 15), this completes the proof. $\square$

P r o o f o f t h e C o r o l l a r y. It suffices to show that Σ fixes the points u,v and the line Y. Suppose first that $v^\Sigma\neq v$. Then Σ has Lenz type at least IV (see [21, (64.18) and § 24]), and $\mathcal{P}\cong\mathcal{O}$ by [21, (81.19)]. Hence $v^\Sigma=v$ and, dually, $W^\Sigma=W$. If $Y^\Sigma\neq Y$, the action of Φ shows that Y^Σ is open in the pencil $\mathfrak{L}_v$, see [21, (96.25)]. Consequently, $\dim\Sigma=\dim\Sigma_Y+8\geqq\dim\Delta+8\geqq 44$. Moreover, a^Σ is open in P since T is transitive on Y. There is a reflection α with center a in Φ, and [21, (61.20)] implies that $\mathcal{P}$ is a translation plane. Again $\mathcal{P}$ is classical by [21, (81.19)]. Dually, it follows that $u^\Sigma=u$. $\square$

In the next theorem, all planes admitting a connected group Δ such that $\dim\Delta\geqq 33$ and Δ fixes exactly two points and two lines will be described by coordinate methods. Remember from [8, VI 3], [5, XI.4.2] or [21, (24.4)] that an affine plane with a transitive group of 'vertical' translations can be coordinatized by a so-called[1)] Cartesian field $(K,+,\cdot)$. This means that $(K,+)$ is a group and that each non-vertical line is given by an equation $y=s\cdot x+t$. The other algebraic properties of a Cartesian field just express the fact that these lines together with the vertical ones indeed yield an affine plane. In the situation considered here, the additive group of K may be identified with $(\mathbb{O},+)$ and multiplication is continuous. According to [21, (43.6)], this suffices for the projective closure of the affine plane over K to be a compact topological plane.

1) Several authors write *Cartesian group* for a linear ternary field with associative addition even though such a structure is like a ring rather than a group.

Theorem 2. *Let* $(\mathbb{R}, +, *, 1)$ *be a topological Cartesian field with unit element* 1 *and assume that* $(-r) * s = -r * s = r * (-s)$ *holds identically. Define a new multiplication on the octonion algebra* $(\mathbb{O}, +,)$ *by*

$$a \circ x = |a| * |x| \, (|a| \, |x|)^{-1} a x \quad for \quad a, x \neq 0 \quad and \quad 0 \circ x = a \circ 0 = 0.$$

Then $(\mathbb{O}, +, \circ)$ *is a topological Cartesian field. A plane* $\mathcal{P}$ *can be coordinatized by such a Cartesian field if and only if* $\mathcal{P}$ *satisfies the hypotheses of Theorem* 1.

Remarks. 1) An analogous construction can be applied to $\mathbb{C}$ and to $\mathbb{H}$ instead of $\mathbb{O}$.

2) The Hurwitz ternary fields of Plaumann and Strambach [10] are isomorphic to certain of the Cartesian fields defined in Theorem 2. In [10] a new addition $\oplus$ is introduced on $\mathbb{K} \in \{\mathbb{C}, \mathbb{H}, \mathbb{O}\}$ by means of a suitable homeomorphism $\varrho : [0, \infty) \approx [0, \infty)$ and the associated homeomorphism $\hat{\varrho} : \mathbb{K} \approx \mathbb{K}$, where $\hat{\varrho}(x) = \varrho(|x|) \, |x|^{-1} x$ for $x \neq 0$ and $\hat{\varrho}(0) = 0$. The *radial distorsion* of the ordinary addition of $\mathbb{K}$ is given by $\hat{\varrho}(a \oplus b) = \hat{\varrho}(a) + \hat{\varrho}(b)$. Reversing this transformation yields a topological Cartesian field $(\mathbb{K}, +, \circ)$ with

$$a \circ b = \hat{\varrho}\,(\hat{\varrho}^{-1}(a)\,\hat{\varrho}^{-1}(b)) = \varrho(\varrho^{-1}(|a|)\varrho^{-1}(|b|))\,(|a||b|)^{-1}\,ab \text{ for } a, b \neq 0,$$

corresponding to the real Cartesian field $(\mathbb{R}, +, *)$, where

$$r * s = \operatorname{sgn} r \cdot \operatorname{sgn} s \cdot \varrho\,(\varrho^{-1}(|r|)\,\varrho^{-1}(|s|)).$$

Such a representation is possible if, and only if, the multiplication $*$ is associative. (Note that any *group* $((0, \infty), *)$ is isomorphic as a topological group to the ordinary group of positive real numbers.)

Proof of Theorem 2. **A)** Suppose first that $\mathcal{P}$ has the properties of Theorem 1.

1) If $\mathcal{P}$ is coordinatized with respect to the quadrangle $0 = a, u, v, e$ in the usual way (as in [21, § 22]), then, by the remarks above, the coordinate structure is a Cartesian field $(\mathbb{O}, +, \circ)$.

2) According to Theorem 1, there is a group $\Phi \cong \mathrm{Spin}_8\mathbb{R}$ in the stabilizer of the triangle a, u, v, and Φ induces on $\mathsf{T} \lhd \Delta$ the group $\mathrm{SO}_8\mathbb{R}$. Consequently, $\mathsf{T} \cong \mathbb{R}^8$ is commutative.

3) For $c \in Y \setminus \{a, v\}$, it follows with ($\square$) that $\Phi_c \cong \mathrm{Spin}_7\mathbb{R}$ has orbits homeomorphic to $\mathrm{Spin}_7/\mathrm{G}_2 \approx \mathbb{S}_7$ on W as well as on au. By [12, Th. a], the group Φ acts linearly on all 3 sides of the fixed triangle. The center of Φ contains 3 reflections with axes au, av, and uv, and Φ induces on each of these lines a group $\mathrm{SO}_8\mathbb{R}$.

4) The reflections with centers a or v invert each translation in T. Hence the maps in the center of Φ have the form $(x, y) \mapsto (\pm x, \pm y)$. For the multiplication this gives the identity $(-r) \circ s = -(r \circ s) = r \circ (-s)$.

5) The group $\Phi_e \cong \mathrm{G}_2$ centralizes a one-parameter subgroup $\mathsf{E} < \mathsf{T}$. The fixed elements of Φ_e form a 2-dimensional subplane $\mathcal{E}$, and E induces a transitive group of

vertical translations on $\mathcal{E}$. Therefore, $\mathcal{E}$ is coordinatized by a Cartesian field $(\mathbb{R}, +, *)$, in fact, by a Cartesian subfield of $(\mathbb{O}, +, \circ)$, so that $*$ is a restriction of the multiplication $\circ$ and satisfies the identity of Theorem 2.

6) A copy of the group $\Phi \cong \mathrm{Spin}_8\mathbb{R}$ acts in the standard way on the Moufang plane $\mathcal{O}$. This action is discussed in [21, (17.12–16)]. In particular, the *triality* automorphism permutes the 3 central involutions of Φ. Any element of Φ acts on the point set $\mathbb{O}\times\mathbb{O}$ of the affine plane in the form $(x, y) \mapsto (Ax, By)$, where A and B belong to a *triad* (A, B, C) such that $A, B, C \in \mathrm{SO}_8\mathbb{R}$ and identically $B(s{\cdot}x) = Cs{\cdot}Ax$ with respect to the ordinary multiplication of $\mathbb{O}$. There are 3 conjugacy classes of subgroups isomorphic to $\mathrm{Spin}_7\mathbb{R}$; indeed, two such subgroups are conjugate if, and only if, they contain the same central involution of Φ.

7) Because Φ acts linearly on the sides of the triangle a, u, v, the point set of the affine plane $\mathcal{P}^{uv}$ can be identified with $\mathbb{O}\times\mathbb{O}$ in such a way that the sets $\{x\} \times \mathbb{O}$ and $\mathbb{O} \times \{y\}$ are lines of $\mathcal{P}$ and Φ acts as on the affine Moufang plane.

8) Put $w = ae \cap uv$. Up to conjugacy, the elements of $\Phi_w \cong \mathrm{Spin}_7\mathbb{R}$ are given by all triads of the form (A, A, C): we may assume, in fact, that Φ_w fixes the line with the equation $y = x$ in $\mathcal{O}$. This implies $C(1) = 1$ and $B = A \in \mathrm{Spin}_7\mathbb{R}$.

9) There is a homeomorphism $f : \mathbb{O} \to \mathbb{O}$ satisfying $f(0) = 0$ and $f(1) = 1$ such that the affine line ae in $\mathcal{P}$ is the point set $\{(x, f(x)) \mid x \in \mathbb{O}\}$. Since ae is Φ_w-invariant, $Af(x) = f(Ax)$ for all $A \in \mathrm{Spin}_7\mathbb{R}$ and $x \in \mathbb{O}$. The group $\mathrm{Spin}_7\mathbb{R}$ acts transitively on each sphere of constant norm in $\mathbb{O}$; the subgroup $\mathrm{G}_2 = \mathrm{Aut}\,\mathbb{O} \cong \Phi_e$ fixes exactly the reals. It follows that f induces a homeomorphism on $\mathbb{R}$. Writing $x = A|x|$, we see that $f(x) = Af(|x|) = f(|x|)|x|^{-1}x$.

10) Rescale the first coordinates by writing $f(x)$ instead of x. In the new coordinates, the line ae is given by the equation $y = x$. The orbits of Φ on the axes are still spheres of constant norm, but the action of Φ on au need not be linear anymore. As stated at the beginning of the proof, the new coordinates form a Cartesian field $(\mathbb{O}, +, \circ)$. Remember from step 4) that the fixed plane $\mathcal{E}$ of Φ_e is coordinatized by a Cartesian field $(\mathbb{R}, +, *)$. It remains to express the multiplication $\circ$ by $*$.

11) The condition $B(sx) = CsAx$ for triads holds also in the new coordinates, since rescaling affected only the x-coordinate and changed each x by a positive scalar. By [21, (17.6)], the group $\mathrm{Aut}\,\mathcal{O}$ is transitive on quadrangles of the Moufang plane, and the action of Φ on $\mathcal{O}$ shows that each pair of elements $s, x \in \mathbb{O}$ can be represented in the form $s = C|s|$ and $x = A|x|$ for some triad (A, B, C). It follows that $sx = |s||x|B(1)$, and the geometric interpretation of $\circ$ implies $s \circ x = C|s| \circ A|x| = B(|s|\circ|x|) = B(|s|*|x|) = |s|*|x|B(1)$, hence $s \circ x = |s|*|x|(|s||x|)^{-1}sx$, as claimed above.

B) *The construction in Theorem* 2 *always yields a* (topological) *Cartesian field*.

Continuity of $\circ$ being obvious, it suffices to show that the map $x \mapsto a \circ x - b \circ x$ is bijective whenever $a \neq b$; since both factors play symmetric rôles, $s \mapsto s \circ a - s \circ b$

is then also a bijective map of $\mathbb{O}$. We write $x = |x|x_1$ with $|x_1| = 1$ and distinguish 3 cases:

1) $b = 0$. The condition $a \circ x = |a|{*}|x|\, a_1 x_1 = d\ (\neq 0)$ implies $|a|{*}|x| = |d|$. By assumption, $(\mathbb{R}, +, *)$ is a Cartesian field. Hence $|x|$ is uniquely determined by a, d, and then also x from $a_1 x_1 = d_1$. Analogously, $x \mapsto x \circ c$ is bijective for each $c \neq 0$.
2) $a_1 = b_1$ (and then $|a| \neq |b|$). Using the last part of case 1), we may assume that $a \circ x - b \circ x = d \neq 0$. It follows that $|a|{*}|x| - |b|{*}|x| = |d|$. Again $|x|$ ist uniquely determined, and $x_1 = a_1^{-1} d_1$.
3) $a_1 \neq b_1$ (and $|a| \geqq |b|$). Let $a \circ x - b \circ x = d \neq 0$, put $h = a_1 \overline{b_1} + b_1 \overline{a_1} = 2\,\mathrm{Re}\, a_1 b_1^{-1}$, and note that $a_1 \neq b_1$ implies $h < 2$. We obtain
$$d\overline{d} = (|a|{*}|x|)^2 + (|b|{*}|x|)^2 - (|a|{*}|x|)(|b|{*}|x|)\, h := q(|x|).$$
Because $(\mathbb{R}, +, *)$ is a Cartesian field,
$$r \mapsto q(r) = (|a|{*}r - |b|{*}r)^2 + (2 - h)(|a|{*}r)(|b|{*}r)$$
defines a strictly increasing bijection of $[0, \infty)$. Hence $q(r) = |d|^2$ has a unique solution, and x is given by $|x| = r$ and $((|a|{*}r)a_1 - (|b|{*}r)b_1)x_1 = d$.

C) *If the plane $\mathcal{P}$ is coordinatized by a Cartesian field $(\mathbb{O}, +, \circ)$, where multiplication is defined as in Theorem* 2, *then* $\mathrm{Aut}\,\mathcal{P}$ *has a subgroup* $\Phi \cong \mathrm{Spin}_8\mathbb{R}$, *and* Φ *fixes a triangle.*

In fact, each triad (A, B, C) with respect to the ordinary multiplication of $\mathbb{O}$ satisfies
$$B(s \circ x) = |s|{*}|x|\ Cs_1{\cdot}Ax_1 = |Cs|{*}|Ax|\ (Cs)_1(Ax)_1 = Cs \circ Ax.$$
Hence the maps $(x, y) \mapsto (Ax, By)$ are automorphisms of the affine plane $\mathcal{P}^W$. They form a group $\Phi \cong \mathrm{Spin}_8\mathbb{R}$. Obviously, Φ fixes the coordinate axes. □

Remark. If $\mathcal{P}$ is not classical, then $\dim\Delta \leqq 37$ by Theorem 1. The case $\dim\Delta = 37$ can be described more precisely. As in step A5), we consider a subgroup $\Lambda \cong \mathrm{G}_2$ of Φ, the fixed plane $\mathcal{E}$ of Λ, the 'vertical' translation group $\mathsf{E} = \mathsf{T} \cap \mathrm{Cs}\,\Lambda$, and the coordinatizing Cartesian field $(\mathbb{R}, +, *)$ of $\mathcal{E}$. The reflection in Φ with center u will be denoted by σ. If $\dim\Delta = 37$, then the connected group $\triangledown = \Delta_a$ is a direct product $\mathsf{E}{\times}\Phi$ with $\mathsf{E} = \sqrt{\triangledown} \cong \mathbb{R}$, and P is either a group of homologies of $\mathcal{P}$ with axis av or P induces on $\mathsf{T} \cong \mathbb{R}^8$ a group of homotheties (because Φ acts irreducibly on T). Each element of P acts on the affine plane $\mathcal{E}^{uv}$ in the form $(\xi, \eta) \mapsto (\alpha(\xi), r\eta)$, where $\alpha : \mathbb{R} \to \mathbb{R}$ is a homeomorphism fixing 0 and $r > 0$. A map of this form is a collineation of $\mathcal{E}$ if, and only if, $r(s * x) = s' * \alpha(x)$ for some s'. It then extends to an element of $\mathrm{Cs}_\Delta\Phi$. (Put $\widetilde{\alpha}(x) = \alpha(|x|)x_1$. Using the definition of $\circ$, it can easily be verified that $(x, y) \mapsto (\widetilde{\alpha}(x), ry)$ is indeed an automorphism of $\mathcal{P}$ which commutes with Φ.) It remains to determine those planes $\mathcal{E}$ which admit a 2-dimensional group EP. This has been accomplished by GROH and others, see [21, (38.5)].

Digression. Suppose that $\dim\Delta > 37$. Then $\dim\Delta = 38$ by the stiffness theorem (□), and $\mathsf{E}(\bigtriangledown \cap \mathrm{Cs}\,\Lambda)$ induces on $\mathcal{E}$ a 3-dimensional group. Such planes are known explicitly, cf. [21, § 37]. If $\mathcal{E}$ is not classsical, then $\mathrm{Aut}\,\mathcal{E}$ does not contain two commuting reflections ([21, (37.6)]), but the center of Φ does. Consequently, $\mathcal{E}$ is classical, $(\mathbb{R}, +, *)$ is a field, and $\circ$ coincides with the ordinary multiplication of the octonions. This shows again that $\dim\Delta \geqq 38$ implies $\mathcal{P} \cong \mathcal{O}$.

We return to non-classical planes with $\dim\Delta = 37$. There are two cases:

(I) If $\mathsf{EP} \cong \mathbb{R}^2$, then P fixes a^{E} pointwise and $\mathsf{P}\langle\sigma\rangle$ is a transitive group of homologies of $\mathcal{E}$. Hence, the multiplication $*$ is associative and $\mathcal{P}$ is a plane over a Hurwitz ternary field as discussed in the remarks following Theorem 2, see [4] or [11, 2.7.11.3] for a geometric description of the corresponding planes $\mathcal{E}$.

(II) $\mathsf{EP} \cong \mathrm{L}_2$. The points u and v and the lines av and uv are fixed by EP. All further fixed points must lie on uv, all further fixed lines must pass through v. There cannot be a further fixed point and a further fixed line at the same time, since then P would fix a quadrangle of $\mathcal{E}$, which contradicts [21, (32.10)]. The planes $\mathcal{E}$ where EP has just two fixed points and two fixed lines have been determined by I. Schellhammer, see [11, 2.7.11.4]. The case where EP has more than two fixed points (and consequently just two fixed lines) has been dealt with by H.-J. Pohl, see [11, 2.7.11.5]. If EP has more than two fixed lines, then $\mathcal{E}$ is dual to one of the planes determined by Pohl.

A typical class of examples for the planes found by Schellhammer may be described as follows: Let f be a homeomorphism of $[0,\infty)$. Put $h(x) = \int_0^x f(t)dt$ and $h(-x) = -h(x)$ for $x \leqq 0$. Then the lines of $\mathcal{E}^{uv}$ are the parallels to the axes, the set $L = \{(x, h(x)) \mid x \in \mathbb{R}\}$ and all its images under the maps $(x, y) \mapsto (\pm\, ax, ay+b)$ with $a > 0, b \in \mathbb{R}$.

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